Linear Complementarity Systems and Cone-Copositive Lyapunov Stability

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***Abstract*—Exponential stability of the origin of linear complementarity systems (LCS) is analyzed by applying Lyapunov theory. By representing the feasibility and the solution sets of the LCS as cones, a cone-copositive approach is used to get sufficient stability conditions expressed in terms of linear matrix inequalities (LMI). The proposed method is constructive in the sense that the solution of the set of LMI directly provides a quadratic Lyapunov function. Sufficient conditions for piecewise quadratic Lyapunov functions are obtained, as well. Illustrative examples show the effectiveness of the approach.**

***Index Terms*—Stability of hybrid systems, Lyapunov methods, Hybrid systems, Switched systems, LMIs.**

I. Introduction

LINEAR complementarity systems (LCS) are characterized by continuous-time linear time-invariant dynamics coupled with input-output variables constrained by complementarity conditions [1]. Structural properties for LCS have been investigated, such as controllability, observability, passivity [2] and Zeno behaviour [3].

The exponential stability problem of the origin of LCS is considered herein. This issue has been analyzed in [4] where a quadratic Lyapunov function of both state and complementarity variables has been used in order to prove the stability of the origin. When specified in terms of the state variables alone, the Lyapunov function becomes piecewise quadratic. The sufficient stability conditions proposed in that paper are in general not easy to verify, moreover the state solution of the LCS is assumed to be of class C^1. A similar Lyapunov approach has been used in [5] where the authors extended the stability analysis to the presence of delays.

The stability issue can be also tackled by exploiting a cone-copositive approach, as in [6] where asymptotic stability conditions are obtained for evolution variational inequalities, which include some particular classes of LCS. Cone-copositive programming, applied to unknown matrices, has been used to achieve LMI which express sufficient conditions for the asymptotic stability of conewise linear systems [7] and piecewise affine systems with polyhedral state space partitions [8]. In this paper, by looking for constructive and less conservative conditions, we represent the solution set of the complementarity problem as the union of polyhedral cones and then we use the cone-copositive approach for obtaining operative LMI which synthesize stability conditions with a common quadratic Lyapunov function. A generalization to the case of piecewise quadratic Lyapunov functions defined on a conic partition of the solution set is proposed, too.

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II. Notations and Preliminaries

The notation $\|v\|$ indicates the euclidean norm of the vector $v \in \mathbb{R}^n$. A set $\mathcal{C} \subseteq \mathbb{R}^n$ such that for all $v \in \mathcal{C}$ it is $\alpha v \in \mathcal{C}$ for any scalar $\alpha \geq 0$, is called a cone. Given a finite number ρ of points $\{r_\ell\}_{\ell=1}^{\rho}$, $r_\ell \in \mathbb{R}^n$, $\rho \in \mathbb{N}$, the (convex) set of points $v \in \mathbb{R}^n$ such that $v = \sum_{\ell=1}^{\rho} r_\ell \theta_\ell$, with $\theta_\ell \in \mathbb{R}_+$, $\mathbb{R}_+$ being the set of nonnegative real numbers, is called a *polyhedral cone*. The points $\{r_\ell\}_{\ell=1}^{\rho}$ are called rays of the polyhedral cone. The matrix $R \in \mathbb{R}^{n\times\rho}$ whose columns are the points $\{r_\ell\}_{\ell=1}^{\rho}$ in an arbitrary order is called ray matrix and identifies a $\mathcal{V}$-representation of the polyhedral cone [9]. Any non-empty polyhedral cone can be equivalently defined as the set of points $v \in \mathbb{R}^n$ such that $Ev \geq 0$ with $E \in \mathbb{R}^{\mu\times n}$ which identifies a $\mathcal{H}$-representation of the polyhedral cone [9].

Given a cone $\mathcal{X}$, a collection of polyhedral cones $\{\mathcal{X}_i\}_{i=1}^{N}$ with N positive integer provides a conic polyhedral partition of $\mathcal{X}$ if $\cup_{i=1}^{N}\mathcal{X}_i = \mathcal{X}$ and $(\mathcal{X}_i \backslash \partial\mathcal{X}_i) \cap (\mathcal{X}_j \backslash \partial\mathcal{X}_j) = \emptyset$ for all $i \neq j$. In the following the term partition will refer to a conic polyhedral partition.

The notation $P \succcurlyeq 0$ indicates that P is positive semidefinite. A matrix P is called a P-matrix if all the principal minors of the matrix are positive. Every positive definite matrix, i.e. $P \succ 0$, is in this class but not vice-versa. A symmetric matrix $P \in \mathbb{R}^{n\times n}$ which is positive semidefinite with respect to a cone $\mathcal{C} \subseteq \mathbb{R}^n$, i.e. $v^\top P v \geq 0$ for any $v \in \mathcal{C}$, is cone-copositive with respect to $\mathcal{C}$. A (strictly) cone-copositive matrix will be denoted by ($P \succ^{\mathcal{C}} 0$) $P \succcurlyeq^{\mathcal{C}} 0$. In the particular case $\mathcal{C} = \mathbb{R}^n_+$, a (strictly) cone-copositive matrix is called (strictly) copositive.

The cone-copositivity of a known symmetric matrix P on a polyhedral cone $\mathcal{C}$ can be checked through a sufficient LMI condition.

Lemma 1: ([8]) Let $P \in \mathbb{R}^{n\times n}$ be a symmetric matrix, $R \in \mathbb{R}^{n\times\rho}$ be a ray matrix of the polyhedral cone $\mathcal{C} \subset \mathbb{R}^n$. If there exists a symmetric (entrywise) positive matrix $N \in \mathbb{R}^{\rho\times\rho}$ such that the following LMI

$$R^\top P R - N \succcurlyeq 0 \tag{1}$$

is satisfied, then P is strictly cone-copositive on $\mathcal{C}$.

III. Complementarity and cone-copositivity

A LCS is defined as a combination of a linear system with complementarity constraints:

$$\dot{x} = Ax + Bz \tag{2a}$$
$$w = Cx + Dz \tag{2b}$$
$$w \geq 0,\ z \geq 0 \tag{2c}$$
$$w^\top z = 0 \tag{2d}$$

where $x \in \mathbb{R}^n$ is the state vector, $(w,z) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+$ are the complementarity variables, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in R^{m\times n}$, $D \in \mathbb{R}^{m\times m}$ are given constant matrices. The complementarity conditions (2c)–(2d) mean that w and z are componentwise nonnegative and for each pair of scalar components one of the two must be zero.

A. Feasibility and solution sets related to LCS

For a given x the expressions (2b)–(2d) define the linear complementarity problem $LCP(Cx, D)$. The set of $z \in \mathbb{R}^m$ which satisfy (2b)–(2c) is called the feasibility set of the $LCP(Cx, D)$ [10]. By considering the orthogonality constraint (2d) too, one can define the solution set of the LCP:

$$SOL(Cx, D) = \{z \in \mathbb{R}^m \,|\, w = Cx + Dz, \\ w \geq 0, z \geq 0, w^\top z = 0\}. \tag{3}$$

If D is a P-matrix than $SOL(Cx, D)$ is a singleton for any $x \in \mathbb{R}^n$ [2]. In the more general case of D not being a P-matrix, $SOL(Cx, D)$ is not necessarily singleton and it could also be empty. We are interested to restrict our analysis to the set of x for which $SOL(Cx, D)$ is nonempty.

In particular, in view of the stability result for the origin of the state space of the system (2), the evaluation of $SOL(0, D)$ plays a crucial role. Clearly $0 \in SOL(0, D)$ for all D and this solution is unique when D is a P-matrix. In general, $SOL(0, D)$ could have multiple solutions as shown by the following result.

Lemma 2: The $LCP(0, D)$ with $D = -D^\top$ skew symmetric admits non zero solutions.

Proof: Let us consider the polyhedral cone $\mathcal{K} = \{v \in \mathbb{R}^m \,|\, Dv \geq 0\}$, whose $\mathcal{H}$-representation is given by the matrix D, and the corresponding dual cone $\mathcal{K}^* = \{\omega \in \mathbb{R}^m \,|\, \omega^\top v \geq 0,\ \forall v \in \mathcal{K}\}$. Note that for any polyhedral cone $\{v \,|\, Mv \geq 0\}$ the corresponding dual cone is given by $\{\omega | \omega = M^\top y,\ y \geq 0\}$. Thus, being D a skew symmetric matrix, the dual cone becomes $\mathcal{K}^* = \{\omega \,|\omega = -Dy,\ y \geq 0\}$.

Since D is skew symmetric we have $v^\top Dv = 0\ \forall v \in \mathbb{R}^m$. Thus, from (3) in this particular case $SOL(0, D) = \mathcal{K} \cap \mathbb{R}^m_+$. Now let us assume by contradiction that $\mathcal{K} \cap \mathbb{R}^m_+ = \{0\}$. Since $\mathcal{K}$ and $\mathbb{R}^m_+$ are nonempty convex sets, $\operatorname{int} \mathbb{R}^m_+ \neq \emptyset$ and $\mathcal{K} \cap \operatorname{int} \mathbb{R}^m_+ = \emptyset$ by contradiction, it comes out that $\mathbb{R}^m_+$ and $\mathcal{K}$ can be properly separated (see [11, Theorem 2.39]). Furthermore, by considering that $\mathcal{K}$ and $\mathbb{R}^m_+$ are convex cones, it follows that $-\mathcal{K}^* \cap \mathbb{R}^m_+ \neq \{0\}$ (see [11, Exercise 6.48, (a)]). Thus, there exists a $\bar{\omega} \neq 0$ such that $\bar{\omega} \in \mathbb{R}^m_+$ and $-\bar{\omega} \in \mathcal{K}^*$ and as a result there exists a $\bar{y} \geq 0$ such that $\bar{\omega} = D\bar{y} \geq 0$. The existence of a nonzero $\bar{y} \in \mathcal{K} \cap \mathbb{R}^m_+$, contradicts the hypothesis. ■

If x is a variable, we consider all x and z which satisfy (2b)–(2c). By introducing the vector $\xi \in \mathbb{R}^{n+m}$ the expressions (2b)–(2c) can be rewritten as

$$\Gamma\xi \geq 0 \tag{4}$$

with $\Gamma = \left(\begin{smallmatrix} C & D \\ 0 & I \end{smallmatrix}\right) \in \mathbb{R}^{2m\times(n+m)}$. The set of ξ which satisfy (4) is called the feasibility set related to the LCS (2) and it is a polyhedral cone in $\mathbb{R}^{n+m}$ with the matrix Γ defining its $\mathcal{H}$-representation:

$$\mathcal{F} = \{\xi \in \mathbb{R}^{n+m} \,|\, \Gamma\xi \geq 0\}, \tag{5}$$

i.e., all ξ such that $\Gamma\xi$ is componentwise nonnegative.

We now consider the orthogonality constraint (2d) too. Since w and z are both nonnegative, under (2b)–(2c) the constraint (2d) can be replaced by $2w^\top z \leq 0$ which can be rewritten by means of the quadratic inequality $-\xi^T H\xi \geq 0$ with $H \in \mathbb{R}^{(n+m)\times(n+m)}$ given by

$$H = \begin{pmatrix} 0 & C^\top \\ C & D + D^\top \end{pmatrix}. \tag{6}$$

We call

$$\mathcal{S} = \{\xi \in \mathcal{F} | -\xi^T H\xi \geq 0\} \tag{7}$$

the solution set related to the LCS and it contains all pairs (x, z) satisfying (2). The solution set $\mathcal{S}$ is different (also dimensionally) from the solution set $SOL(Cx, D)$ in (3) which contains all (and only) z satisfying the $LCP(Cx, D)$ for a given x. Clearly $0 \in \mathcal{S}$, i.e. $\mathcal{S}$ is nonempty.

The set $\mathcal{S}$ is itself a (not necessarily convex) cone, as it can be verified by using the cone definition. It is interesting to compare the polyhedral cone $\mathcal{F}$ and the cone $\mathcal{S}$. Clearly $\mathcal{S} \subseteq \mathcal{F}$ by definition. Moreover, since all $\xi \in \mathcal{F}$ correspond to nonnegative w and z, while the set $\mathcal{S}$ includes the condition $w^\top z = 0$, it is $\mathcal{S} \subseteq \partial\mathcal{F}$.

The quadratic form in (7) is a cone-copositive condition on the cone $\mathcal{S}$ for the matrix $-H$, but it is not a cone-copositive condition on $\mathcal{F}$.

B. Cone-copositivity on the solution set

The stability analysis will exploit conditions for the sign of quadratic forms on the solution set $\mathcal{S}$ that is not necessarily convex, differently from the polyhedral cone $\mathcal{F}$. Hence the problem of verifying the sign of a quadratic form $\xi^\top Q\xi$ on $\mathcal{S}$, with $Q \in \mathbb{R}^{(n+m)\times(n+m)}$, i.e. the cone-copositive condition

$$-Q \succ^{\mathcal{S}} 0, \tag{8}$$

cannot be tackled by applying Lemma 1 to (8). From the definition of $\mathcal{S}$ in (7), the condition (8) can be equivalently written with $-\xi^T Q\xi > 0$, $\forall \xi \in \mathcal{F} - \{0\}$ such that $-\xi^T H\xi \geq 0$, i.e. a cone-copositive condition for $-Q$ on $\mathcal{F}$ plus a quadratic constraint with the matrix H. This consideration leads to the following result.

Lemma 3: Let $Q \in \mathbb{R}^{(n+m)\times(n+m)}$ be a symmetric matrix, $R \in \mathbb{R}^{(n+m)\times\rho}$ be a ray matrix of the polyhedral cone $\mathcal{F} \subseteq \mathbb{R}^{n+m}$, $H \in \mathbb{R}^{(n+m)\times(n+m)}$ be a symmetric matrix. If there exist $\{N, \tau\}$ with $N \in \mathbb{R}^{\rho\times\rho}$ symmetric (entrywise) positive matrix and scalar $\tau > 0$, such that the following linear matrix inequality

$$-R^\top (Q - \tau H) R - N \succcurlyeq 0 \tag{9}$$

is satisfied, then (8) holds.

Proof: Since $\mathcal{F}$ is a polyhedral cone, by using Lemma 1, from (9) it follows $-(Q - \tau H) \succ^{\mathcal{F}} 0$. Thus one can write $-\xi^\top Q \xi > -\tau \xi^\top H \xi^\top$, $\forall \xi \in \mathcal{F} - \{0\}$. Being $\mathcal{S} \subseteq \mathcal{F}$, from (7) it comes out $-\xi^\top Q \xi > 0$, $\forall \xi \in \mathcal{S} - \{0\}$ and then (8) is satisfied. ∎

IV. Cone-copositive Lyapunov function

The cone-copositive approach can be used for the stability analysis of LCS. Let us introduce the solution concept for the system of interest. For those x such that $SOL(Cx, D)$ related to (2) is nonempty, by collecting (2)-(3) we can write the differential inclusion [3]

$$\dot{x} \in Ax + B\, SOL(Cx, D). \tag{10}$$

More general formalisms, based on evolution variational inequalities or differential inclusions with maximal monotone operators, can be used for LCS [6], [12]. In this paper, for notation convenience related to copositivity, complementarity and stability conditions expressed through LMI, the differential inclusion form (10) is adopted.

We say that a function $x(t) : \mathbb{R}_+ \to \mathbb{R}^n$ is a Carathéodory solution of (10) on $t \in \mathbb{R}_+$ for the initial state $x(0) = x_0$ if $x(t)$ is absolutely continuous and it satisfies (10) almost everywhere for all $t \in \mathbb{R}_+$. In the following we assume that (10) has a Carathéodory solution for any $x_0 \in \mathbb{R}^n$ such that $(x_0, SOL(Cx_o, D)) \in \mathcal{S}$. Moreover we assume that the system does not exhibit Zeno behaviors. A sufficient condition for the existence and uniqueness of solution is that $B\, SOL(Cx, D)$ is a singleton for all $x \in \mathbb{R}^n$ [4, Prop. 2.1], or that some structural properties like passivity hold [2], [13].

Clearly the origin is an equilibrium point for (10). We say that the origin is asymptotically stable if it is stable and all solutions of (10) converge to the origin for any $x_0 \in \mathbb{R}^n$ such that $(x_0, SOL(Cx_o, D)) \in \mathcal{S}$. If the convergence is exponential we say that the origin is exponentially stable.

The stability analysis of (10) could be tackled by exploiting the general result of Theorem 4.1 in [14] which concludes the asymptotic stability through Lyapunov functions sign conditions holding on the entire state space. This approach could be conservative. By restricting the sign analysis to the solution set $\mathcal{S}$, one could expect less conservative results.

Let us consider as a candidate Lyapunov function

$$V(x) = x^\top P x \tag{11}$$

with $P \in \mathbb{R}^{n \times n}$ symmetric matrix. The following Lemma provides stability conditions in terms of LMI.

Lemma 4: Consider the differential inclusion (10) and say $R \in \mathbb{R}^{(n+m) \times \rho}$ a ray matrix of the polyhedral cone (5). The origin is exponentially stable if there exist $\{P, N, \tau\}$ with $P \in \mathbb{R}^{n \times n}$, $P \succ 0$, $N \in \mathbb{R}^{\rho \times \rho}$ entrywise positive matrix, a scalar $\tau > 0$, such that

$$-R^\top (Q_P - \tau H) R - N \succcurlyeq 0 \tag{12}$$

is satisfied with $H \in \mathbb{R}^{(n+m) \times (n+m)}$ given by (6) and $Q_P \in \mathbb{R}^{(n+m) \times (n+m)}$ given by

$$Q_P = \begin{pmatrix} A^\top P + PA & PB \\ B^\top P & 0 \end{pmatrix}. \tag{13}$$

Proof: Let $\{P, N, \tau\}$ be a solution of (12). Since $P \succ 0$, the quadratic function (11) is strictly positive and satisfies

$$\lambda_{\min} \|x\|^2 \leq V(x) \leq \lambda_{\max} \|x\|^2 \tag{14}$$

for $x \in \mathbb{R}^n$, where $\lambda_{\min}$ and $\lambda_{\max}$ are the (positive) minimum and the maximum eigenvalues of P.

We want to analyze the sign, on the set $\mathcal{S}$, of the following scalar product

$$\langle \nabla_x V(x),\, Ax + Bz \rangle = \xi^\top Q_P \xi, \tag{15}$$

with $\xi = \left(\begin{smallmatrix} x \\ z \end{smallmatrix} \right)$, where Q_P is given by (13). From Lemma 3 applied to (12) it follows that $-\xi^\top Q_P \xi > 0, \forall \xi \in \mathcal{S}$. As a result, being $\mathcal{S}$ a cone, one can write $-\xi^\top Q_P \xi \geq c_M \|\xi\|^2$, $\forall \xi \in \mathcal{S}$ where $c_M = \min\{-\xi^\top Q_P \xi \,|\, \xi \in \mathcal{S},\, \|\xi\| = 1\}$. Thus, from the last inequality and by considering that $\|\xi\|^2 \geq \|x\|^2$, we obtain $\langle \nabla_x V(x),\, Ax + Bz \rangle \leq -c_M \|x\|^2, \forall z \in SOL(Cx, D)$. By looking at the differential inclusion (10) and by applying the last inequality and Theorem 4.1 in [14] to any solution $x(t)$ of (10), it follows that

$$V(x(t_2)) - V(x(t_1)) \leq -\int_{t_1}^{t_2} c_M \|x(\sigma)\|^2 \mathrm{d}\sigma, \tag{16}$$

with $t_2 \geq t_1$. By using standard arguments, by combining (14) and (16) and by applying the integral version of the Grönwall–Bellman Lemma, it easily follows that

$$\|x(t)\| \leq \frac{\lambda_{\max}}{\lambda_{\min}} e^{-\frac{c_M}{\lambda_{\max}} t} \|x(0)\| \tag{17}$$

proving the exponential stability of the origin of (10). ∎

The condition $P \succ 0$ can be relaxed by considering a cone-copositive condition of $P \in \mathbb{R}^{n \times n}$ on the projection of $\mathcal{F} \subseteq \mathbb{R}^{n+m}$ to the state space of (10), say $\mathcal{F}_x$. To this aim consider the following partition of a ray matrix $R \in \mathbb{R}^{(n+m) \times \rho}$ of the polyhedral cone $\mathcal{F}$:

$$R = \begin{pmatrix} R_x^\top & R_z^\top \end{pmatrix}^\top \tag{18}$$

where $R_x \in \mathbb{R}^{n \times \rho}$, $R_z \in \mathbb{R}^{m \times \rho}$ and $\mathcal{F}_x = \left\{ \xi_x \in \mathbb{R}^n | \xi_x = R_x \theta,\ \theta \in \mathbb{R}_+^\rho \right\}$. The polyhedral cone $\mathcal{F}_x$ contains any solution of (10) being the cone $\mathcal{F}$ the feasibility set related to the LCS (2). We are now ready to prove the following stability theorem.

Theorem 5: Consider the differential inclusion (10) and say $R \in \mathbb{R}^{(n+m) \times \rho}$ a ray matrix of the polyhedral cone (5) partitioned as in (18). The origin is exponentially stable if there exist $\{P, M, N, \tau\}$ with $P \in \mathbb{R}^{n \times n}$ symmetric matrix, $M \in \mathbb{R}^{\rho \times \rho}$ and $N \in \mathbb{R}^{\rho \times \rho}$ symmetric entrywise positive matrices, a scalar $\tau > 0$, such that

$$R_x^\top P R_x - M \succcurlyeq 0 \tag{19a}$$

$$-R^\top (Q_P - \tau H) R - N \succcurlyeq 0 \tag{19b}$$

with H given by (6) and Q_P given by (13).

Proof: Clearly the origin of $\mathbb{R}^n$ belongs to the polyhedral cone $\mathcal{F}_x$. Let $\{P, M, N, \tau\}$ be a solution of (19). Let us consider the candidate Lyapunov function (11) which is strictly positive on the cone $\mathcal{F}_x$ because of (19a) and Lemma 1. It follows that (14) holds with $\lambda_{\min} = \min\{x^\top P x \,|\, x \in \mathcal{F}_x,\, \|x\| = 1\}$ and $\lambda_{\max} = \max\{x^\top P x \,|\, x \in \mathcal{F}_x,\, \|x\| = 1\}$. Then the function (11) is strictly positive along any solution of (10). From (19b) one can write $-\xi^\top Q_P \xi + \tau \xi^\top H \xi - \theta^\top N \theta \geq 0$, $\forall \xi \in \mathcal{F}$, $\forall \theta \in \mathbb{R}^\rho_+$. Then it is $-\xi^\top Q_P \xi \geq \theta^\top N \theta > 0$, $\forall \xi \in \mathcal{S}$, $\forall \theta \in \mathbb{R}^\rho_+$, and by following analogous steps as in Lemma 4 the proof is complete. ∎

Without loss of generality one can choose $\tau = 1$ in Theorem 5 and the result below directly follows.

Corollary 6: Consider the differential inclusion (10). Say $R \in \mathbb{R}^{(n+m)\times\rho}$ a ray matrix of the cone (5) partitioned as in (18). If there exists $P = P^\top \in \mathbb{R}^{n\times n}$ such that

$$R_x^\top P R_x \succ^{\mathbb{R}^\rho_+} 0 \tag{20a}$$

$$-\begin{pmatrix} R_x \\ R_z \end{pmatrix}^\top \begin{pmatrix} A^\top P + PA & PB - C^\top \\ B^\top P - C & -D - D^\top \end{pmatrix} \begin{pmatrix} R_x \\ R_z \end{pmatrix} \succ^{\mathbb{R}^\rho_+} 0, \tag{20b}$$

then the origin is exponentially stable.

If (19b), and consequently (20b), cannot be satisfied, e.g. D is skew symmetric, one can formulate (not asymptotic) stability conditions by relaxing N to be entrywise nonnegative in Theorem 5. That would result in a relaxed version of Corollary 6 where stability is guaranteed by replacing (20b) with the corresponding (non strict) copositive condition. Alternatively, exponential stability can be checked through conditions derived by modifying (20b), so as shown below.

Corollary 7: Consider the differential inclusion (10). Say $R \in \mathbb{R}^{(n+m)\times\rho}$ a ray matrix of the cone (5) partitioned as in (18). If there exist $P = P^\top \in \mathbb{R}^{n\times n}$ and $\gamma > 0$ such that

$$R_x^\top P R_x \succ^{\mathbb{R}^\rho_+} 0 \tag{21a}$$

$$-\begin{pmatrix} R_x \\ R_z \end{pmatrix}^\top \begin{pmatrix} A^\top P + PA + \gamma I & PB - C^\top \\ B^\top P - C & -D - D^\top \end{pmatrix} \begin{pmatrix} R_x \\ R_z \end{pmatrix} \succeq^{\mathbb{R}^\rho_+} 0, \tag{21b}$$

then the origin is exponentially stable.

Proof: From (21b) and by using arguments similar to Corollary 6 one can write $-\xi^\top Q_P \xi - \gamma \xi^\top \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right) \xi \geq 0$, $\forall \xi \in \mathcal{S}$. Then by using (15) and the last inequality one obtains $\langle \nabla_x V(x),\, Ax + Bz \rangle \leq -\gamma \|x\|^2$, for all $z \in SOL(Cx, D)$. Then, by following similar steps of Theorem 5 one concludes the exponential stability of the origin. ∎

Remark 8: Relations between passivity and stability have been investigated in [2] for LCS and in [12] for Lur'e systems. In [6] Sec. VII the assumptions $D = 0$ and $PB = C^\top$ are exploited for achieving copositivity conditions. Our analysis does not require those assumptions and formulates copositive conditions as LMI. The expressions (20b) in Corollary 6 and (21b) in Corollary 7 can be interpreted as a passivity condition and a strict passivity condition [15], respectively, restricted to the feasibility cone.

V. Piecewise cone-copositive Lyapunov function

The stability conditions derived in the previous section are expressed through LMI where a common quadratic Lyapunov function for the LCS is considered. In this section we propose conditions for the existence of a piecewise quadratic Lyapunov function (PWQ-LF) [16] for searching a LF when the LMI of Theorem 5 do not provide a solution.

A. PWQ-LF with D P-matrix

In the case that $D \in \mathbb{R}^{m\times m}$ is a P-matrix one can rewrite (10) in the form of a conewise linear system [3], being $SOL(Cx, D)$ a singleton for all $x \in \mathbb{R}^n$. Let us define the set $\mathcal{M} = \{1, \ldots, m\}$ and say $2^{\mathcal{M}}$ its power set. For each set of indices $\alpha \in 2^{\mathcal{M}}$ and the corresponding relative complement $\bar{\alpha} = \mathcal{M} - \alpha$, consider the two equality conditions $(Cx + Dz)_\alpha = 0$ and $z_{\bar{\alpha}} = 0$, together with the two non strict inequality conditions $(Cx + Dz)_{\bar{\alpha}} \geq 0$ and $z_\alpha \geq 0$. Moreover, define the matrices $C_{\alpha\bullet}$ and $D_{\alpha\bullet}$ as the submatrices obtained from C and D, respectively, by selecting the α rows and all columns. For each α it is

$$(Cx + Dz)_\alpha = C_{\alpha\bullet} x + D_{\alpha\alpha} z_\alpha + D_{\alpha\bar{\alpha}} z_{\bar{\alpha}} \tag{22a}$$

$$(Cx + Dz)_{\bar{\alpha}} = C_{\bar{\alpha}\bullet} x + D_{\bar{\alpha}\alpha} z_\alpha + D_{\bar{\alpha}\bar{\alpha}} z_{\bar{\alpha}}. \tag{22b}$$

By using the definition of the set α and the P-matrix property of D, from (22a) since $z_{\bar{\alpha}} = 0$ one obtains

$$z_\alpha = -(D_{\alpha\alpha})^{-1} C_{\alpha\bullet} x \tag{23a}$$

$$(Cx + Dz)_{\bar{\alpha}} = C_{\bar{\alpha}\bullet} x + D_{\bar{\alpha}\alpha} z_\alpha \tag{23b}$$

for all $\alpha \in 2^{\mathcal{M}}$. By using (23a) in (23b) we can define a polyhedral cone $\mathcal{X}_\alpha$ with the following $\mathcal{H}$-representation

$$\mathcal{X}_\alpha = \{x \in \mathbb{R}^n : \Phi_\alpha x \geq 0\} \tag{24}$$

with $\Phi_\alpha \in \mathbb{R}^{m\times n}$ given by

$$\Phi_\alpha = \begin{pmatrix} -(D_{\alpha\alpha})^{-1} C_{\alpha\bullet} \\ C_{\bar{\alpha}\bullet} - D_{\bar{\alpha}\alpha}(D_{\alpha\alpha})^{-1} C_{\alpha\bullet} \end{pmatrix}. \tag{25}$$

The vector z whose components $z_\alpha \geq 0$ are given by (23a) and $z_{\bar{\alpha}} = 0$, is a solution of the $LCP(Cx, D)$ for all $x \in \mathcal{X}_\alpha$, by construction. Since D is a P-matrix this is the unique solution of the LCP for a given $x \in \mathcal{X}_\alpha$. The cases $\alpha = \emptyset$ and $\alpha = \mathcal{M}$ are included in (24)–(25). Some $\mathcal{X}_\alpha$ can coincide, but the set of distinct cones $\mathcal{X}_\alpha$ defines a partition of the state space as proved by the following result.

Lemma 9: The set of distinct polyhedral cones $\mathcal{X}_\alpha$ with $\alpha \in 2^{\mathcal{M}}$ defined by (24)–(25) provides a partition of $\mathbb{R}^n$.

Proof: First note that $\cup_{\alpha\in 2^{\mathcal{M}}} \mathcal{X}_\alpha = \mathbb{R}^n$. Indeed, let us assume there exists some $\tilde{x} \in \mathbb{R}^n$ with $\tilde{x} \notin \mathcal{X}_\alpha$, $\forall \alpha$. Since D is a P-matrix, there exists a unique solution $\tilde{z}$ of the $LCP(C\tilde{x}, D)$ and by following the construction procedure described above it is easy to get the existence of a corresponding $\mathcal{X}_\alpha$ that contradicts the hypothesis.

We now prove that the intersection of any pair of interiors of $\mathcal{X}_\alpha$ is empty. Note that for all pairs involving at least one degenerate $\mathcal{X}_\alpha$ the condition is trivially satisfied. Then let us consider any α_1 and α_2 such that $\mathcal{X}_{\alpha_1} \neq \mathcal{X}_{\alpha_2}$ and they are not degenerate. Say $\tilde{\mathcal{X}}_{\alpha_1}$ and $\tilde{\mathcal{X}}_{\alpha_2}$ the corresponding open cones characterized by $w_{\alpha_1} = 0$, $z_{\alpha_1} = -(D_{\alpha_1\alpha_1})^{-1} C_{\alpha_1\bullet} x > 0$, $z_{\bar{\alpha}_1} = 0$, $w_{\bar{\alpha}_1} = (C_{\bar{\alpha}_1\bullet} - D_{\bar{\alpha}_1\alpha_1}(D_{\alpha_1\alpha_1})^{-1} C_{\alpha_1\bullet})x > 0$ and analogously for α_2, i.e. $w_{\alpha_2} = 0$, $z_{\alpha_2} > 0$, $z_{\bar{\alpha}_2} = 0$, $w_{\bar{\alpha}_2} > 0$. By contradiction we assume that there exists some $\tilde{x} \in \tilde{\mathcal{X}}_{\alpha_1} \cap \tilde{\mathcal{X}}_{\alpha_2} \neq \emptyset$. For such a $\tilde{x}$ we will get two corresponding solutions

$z_{\bar{\alpha}_1} \neq z_{\bar{\alpha}_2}$. That contradicts the property that the solution of the LCP is a singleton for all $x \in \mathbb{R}^n$ and thus $\mathcal{X}_{\alpha_1}$ and $\mathcal{X}_{\alpha_2}$ can only share a boundary, by construction. ■

When D is a P-matrix, the right-hand side of (10) is a singleton and the differential inclusion can be rewritten as a continuous ordinary differential equation in the conewise linear form

$$\dot{x} = A_\alpha x, \quad x \in \mathcal{X}_\alpha, \quad \forall \alpha \in 2^{\mathcal{M}} \tag{26}$$

where $A_\alpha = A - B_{\bullet\alpha}(D_{\alpha\alpha})^{-1}C_{\alpha\bullet}$, and from Lemma 9 the set $\{\mathcal{X}_\alpha\}_{\alpha\in 2^{\mathcal{M}}}$ provides a partition of $\mathbb{R}^n$.

Remark 10: The partition $\{\mathcal{X}_\alpha\}_{\alpha\in 2^{\mathcal{M}}}$ could contain non full dimensional polyhedral cones. However such sets are faces of other cones of the same partition. By removing these degenerate cones, one obtains a (still complete) partition of the state space for which all the possible local system dynamics are included, thanks to the uniqueness of the solution. Then, without loss of generality, we can exclude in (26) all cones $\mathcal{X}_\alpha$ that are not full dimensional.

We can now formulate the following stability sufficient condition in terms of constrained LMI, whose solution directly provides a continuous piecewise quadratic Lyapunov function for the complementarity system.

Lemma 11: Consider the linear complementarity system represented as in (10) with D being a P-matrix, $\{\mathcal{X}_\alpha\}_{\alpha\in 2^{\mathcal{M}}}$ a partition of the state space with full dimensional polyhedral cones defined by (24)–(25) and say $R_\alpha \in \mathbb{R}^{n\times\rho_\alpha}$ the ray matrix of a $\mathcal{V}$-representation of the cone $\mathcal{X}_\alpha$. Then the origin is exponentially stable if there exist $\{P_\alpha, M_\alpha, N_\alpha\}$ with $P_\alpha \in \mathbb{R}^{n\times n}$ symmetric matrix, $M_\alpha \in \mathbb{R}^{\rho_\alpha\times\rho_\alpha}$ and $N_\alpha \in \mathbb{R}^{\rho_\alpha\times\rho_\alpha}$ entrywise positive matrices such that

$$R_\alpha^\top P_\alpha R_\alpha - N_\alpha \succcurlyeq 0 \tag{27a}$$

$$-R_\alpha^\top \mathcal{Q}_{P_\alpha} R_\alpha - M_\alpha \succcurlyeq 0 \tag{27b}$$

with $\mathcal{Q}_{P_\alpha} = A_\alpha^\top P_\alpha + P_\alpha A_\alpha$, are satisfied together with

$$R_{\alpha_i\alpha_j}^\top (P_{\alpha_i} - P_{\alpha_j}) R_{\alpha_i\alpha_j} = 0 \tag{28}$$

for all pairs (α_i, α_j), with $\alpha_i, \alpha_j \in 2^{\mathcal{M}}$, such that R_{α_i} and R_{α_j} have some common rays and $R_{\alpha_i\alpha_j}$ is the matrix whose columns are the common columns between R_{α_i} and R_{α_j}.

Proof: By Lemma 9 and Remark 10 we can consider the conewise linear system (26). Choose the PWQ function $V(x) = x^\top P_\alpha x$, $x \in \mathcal{X}_\alpha$, $\alpha \in 2^{\mathcal{M}}$, as a candidate Lyapunov function. From (28) it follows that $V(x)$ is continuous across the cones boundaries and then it is continuous in the whole state space. From (27a) and by using Lemma 1 it follows that $V(x)$ is strictly positive for all $x \in \mathbb{R}^n - \{0\}$. From (27b) and by using the analogous of Lemma 1 obtained by substituting P_α with $-(A_\alpha^\top P_\alpha + P_\alpha A_\alpha)$ it follows that $\dot{V}(x(t))$ is strictly negative for all $x \in \mathbb{R}^n - \{0\}$. By considering the finite partitioning of $\mathbb{R}^n$ and that for any $\mathcal{X}_\alpha$ the quadratic functions $x^\top P_\alpha x$ and $-x^\top \mathcal{Q}_{P_\alpha} x$ are strictly cone copositive, following the same arguments used in proving Lemma 4 and Theorem 5, it is possible to show that there exist $c_0, c_1, c_2 > 0$ such that $c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$ and $\dot{V}(x(t)) \leq -c_0 V(x(t))$. From the former inequalities it is clear that $V(x)$ is radially unbounded. Moreover, from the last condition and by following standard Lyapunov arguments it follows that any solution $x(t)$ of (26) goes exponentially to zero, which completes the proof. ■

B. PWQ-LF with D not P-matrix

When D is not a P-matrix, the well-posedness of the LCS (2) is not anymore guaranteed, in general. Conditions for the existence and uniqueness of solution in this case hold under some structural assumptions on the LCS, see [2], [12], [17]. When it is not possible to write the system in the form (26) with the polyhedral partition given by the cones in (24)-(25), the stability problem of the origin of the system (10) is well posed provided that solutions of the differential inclusion exist.

A stability condition based on a continuous PWQ-LF can be obtained by using a polyhedral partition of the cone $\mathcal{S} \subseteq \partial\mathcal{F}$. Consider a polyhedral partition $\{\mathcal{S}_\alpha\}_{\alpha\in 2^{\mathcal{M}}}$ and adopt an approach similar to the case of D being a P-matrix. Consider the two equality conditions $(Cx + Dz)_\alpha = 0$ and $z_{\bar{\alpha}} = 0$, together with the two non strict inequality conditions $(Cx + Dz)_{\bar{\alpha}} \geq 0$ and $z_\alpha \geq 0$ with $\alpha \in 2^{\mathcal{M}}$, $\mathcal{M} = \{1, \ldots, m\}$, $\bar{\alpha} = \mathcal{M} - \alpha$. The feasibility set $\mathcal{F}$ is defined by (4)–(5), i.e. $(Cx + Dz) \geq 0$ and $z \geq 0$. The zero components selected by α can be represented by adding the inequalities $C_{\alpha\bullet}x + D_{\alpha\bullet}z \leq 0$ and $z_{\bar{\alpha}} \leq 0$. Then, one can define the polyhedral cone $\mathcal{S}_\alpha$ with the $\mathcal{H}$-representation

$$\mathcal{S}_\alpha = \{\xi \in \mathbb{R}^{n+m} \,|\, \Gamma_\alpha \xi \geq 0\} \tag{29}$$

where $\Gamma_\alpha = \begin{pmatrix} \Gamma \\ \Delta_\alpha \end{pmatrix}$ with $\Delta_\alpha = \begin{pmatrix} -C_{\alpha\bullet} & -D_{\alpha\bullet} \\ 0 & -I_{\bar{\alpha}\bullet} \end{pmatrix}$, and $I_{\bar{\alpha}\bullet}$ is obtained by selecting the $\bar{\alpha}$ rows and all the columns of the identity matrix. The set of distinct polyhedral cones $\{\mathcal{S}_\alpha\}_{\alpha\in 2^{\mathcal{M}}}$ defined by (29) provides a partition of $\mathcal{S}$ because $\cup_{\alpha\in 2^{\mathcal{M}}}\mathcal{S}_\alpha = \mathcal{S}$ by construction and each $\mathcal{S}_\alpha$ is a degenerate cone being $\mathcal{S}$ a degenerate cone in $\mathbb{R}^{n+m}$.

Say $R_\alpha \in \mathbb{R}^{(n+m)\times\rho_\alpha}$ the ray matrix of a $\mathcal{V}$-representation of the cone $\mathcal{S}_\alpha$. For the following matrix partition $R_\alpha = \begin{pmatrix} R_{\alpha,x} \\ R_{\alpha,z} \end{pmatrix}$ where $R_{\alpha,x} \in \mathbb{R}^{n\times\rho_\alpha}$ and $R_{\alpha,z} \in \mathbb{R}^{m\times\rho_\alpha}$, define $\mathcal{S}_{\alpha,x} = \left\{\xi_x \in \mathbb{R}^n | \xi_x = R_{\alpha,x}\theta,\ \theta \in \mathbb{R}_+^\rho\right\}$ and consider the candidate Laypunov function

$$V(x) = x^\top P_\alpha x, \quad x \in \mathcal{S}_{\alpha_x}. \tag{30}$$

Note that the $\cup\mathcal{S}_{\alpha,x} \subseteq \mathbb{R}^n$ corresponds to the projection of the solution set on the state space. We are now ready to formulate the following stability sufficient condition in terms of constrained LMI, whose solution directly provides a continuous PWQ-LF for the complementarity system.

Lemma 12: Consider the differential inclusion (10) and the partititon $\{\mathcal{S}_\alpha\}_{\alpha\in 2^{\mathcal{M}}}$ of the solution set $\mathcal{S}$ with R_α ray matrix of $\mathcal{S}_\alpha$. The origin is exponentially stable if there exist $\{P_\alpha, M_\alpha, N_\alpha\}$ and $\gamma > 0$ with $P_\alpha \in \mathbb{R}^{n\times n}$ symmetric matrices, $M_\alpha \in \mathbb{R}^{\rho_\alpha\times\rho_\alpha}$ entrywise nonnegative matrices and $N_\alpha \in \mathbb{R}^{\rho_{\alpha_x}\times\rho_{\alpha_x}}$ entrywise positive matrices, such that

$$R_{\alpha,x}^\top P_\alpha R_{\alpha,x} - N_\alpha \succcurlyeq 0 \tag{31a}$$

$$-R_\alpha^\top \left(Q_{P_\alpha} + \gamma \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \right) R_\alpha - M_\alpha \succcurlyeq 0 \tag{31b}$$

with

$$Q_{P_\alpha} = \begin{pmatrix} A^\top P_\alpha + P_\alpha A & P_\alpha B \\ B^\top P_\alpha & 0 \end{pmatrix} \tag{32}$$

are satisfied together with the continuity condition

$$R^\top_{(\alpha_i,x)(\alpha_j,x)}(P_{\alpha_i} - P_{\alpha_j})R_{(\alpha_i,x)(\alpha_j,x)} = 0 \qquad (33)$$

for all pairs (α_i, α_j) such that $R_{\alpha_i,x}$ and $R_{\alpha_j,x}$ have some common rays and $R_{(\alpha_i,x)(\alpha_j,x)}$ is the matrix whose columns are the common columns between $R_{\alpha_i,x}$ and $R_{\alpha_j,x}$.

Proof: Conditions (33) ensure the continuity of (30) for all x such that $(x, SOL(Cx, D)) \in \mathcal{S}$. In particular, (30) takes the same values for all x in the possibly overlapping regions between say $\mathcal{S}_{\alpha_i,x}$ and $\mathcal{S}_{\alpha_j,x}$.

From (31a) it follows that the function (30) is strictly positive in each cone $\mathcal{S}_{\alpha,x}$ and therefore for each x such that $(x, SOL(Cx, D)) \in \mathcal{S}$. By considering that $\langle \nabla_x V(x),\, Ax + Bz\rangle = \xi^\top Q_{P_\alpha}\xi$, $\xi = \left(\begin{smallmatrix} x \\ z \end{smallmatrix}\right)$, from (31b) it follows that

$$\langle \nabla_x V(x),\, Ax + Bz\rangle \leq -\gamma\|x\|^2, \qquad (34)$$

for all $x \in \mathcal{S}_{\alpha,x}$, for all $z = SOL(Cx, D)$ and for all $\alpha \in 2^{\mathcal{M}}$.

Thus, by looking at the differential inclusion (10) with the Lyapunov function (30) and by applying (34) and Theorem 4.1 in [14] to any solution $x(t)$ of (10), the exponential stability of the origin of the system (10) is proved. ∎

VI. Examples

Consider Example 3.1 in [4] whose matrices corresponding to the model (2) are $A = 1$, $B = \left(\begin{smallmatrix} 2 & -2 \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$, $D = \left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right)$, and Example 3.2 in [4] whose matrices are $A = -1$, $B = \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, $D = \left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right)$. We were able to find a solution for (12) and then the asymptotic stability of the origin for each system follows from Lemma 4. Note that in our approach, differently from [4], it is not required to enumerate the different modes.

Consider the Example 3.3 in [4] which is shown to not admit a common quadratic Lyapunov function. The matrices (A, B, C, D) of the model (2) are

$$\begin{pmatrix} -5 & -4 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -3 & 0 & 0 \\ -21 & 0 & 0 \\ 0 & 2 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. Since D is a P-matrix in order to prove stability we can use the procedure in Sec. V-A. By applying (25) the following polyhedra $\mathcal{X}_\alpha$ are obtained: $\mathcal{X}_{\{\emptyset\}} = \mathcal{X}_{\{2,3\}} = \{x_1 \in \mathbb{R}_+, x_2 \in \mathbb{R}, x_3 = 0\}$, $\mathcal{X}_{\{1\}} = \mathcal{X}_{\{1,2,3\}} = \{-x_1 \in \mathbb{R}_+, x_2 \in \mathbb{R}, x_3 = 0\}$, $\mathcal{X}_{\{2\}} = \{x_1 \in \mathbb{R}_+, x_2 \in \mathbb{R}, -x_3 \in \mathbb{R}_+\}$, $\mathcal{X}_{\{3\}} = \{x_1 \in \mathbb{R}_+, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}_+\}$, $\mathcal{X}_{\{1,2\}} = \{-x_1 \in \mathbb{R}_+, x_2 \in \mathbb{R}, -x_3 \in \mathbb{R}_+\}$, $\mathcal{X}_{\{1,3\}} = \{-x_1 \in \mathbb{R}_+, x_2 \in \mathbb{R}, x_3 \in \mathbb{R}_+\}$. The ray matrices R_α can be easily derived by choosing $\rho_\alpha = 4$ for all α.

It is not difficult to derive the dynamic matrices of the different modes and then, differently from [4], we were able to obtain a PWQ-LF directly by applying Lemma 11 whose LMI provide the following matrices

$$\begin{pmatrix} 0.4035 & -0.1692 & 0.3763 \\ -0.1692 & 0.6660 & 0.0635 \\ 0.3763 & 0.0635 & 3.6055 \end{pmatrix}, \quad \begin{pmatrix} 3.0101 & 0.1421 & 0.5671 \\ 0.1421 & 0.6660 & -0.0635 \\ 0.5671 & -0.0635 & 3.6055 \end{pmatrix},$$
$$\begin{pmatrix} 0.4035 & -0.1692 & -0.3763 \\ -0.1692 & 0.6660 & -0.0635 \\ -0.3763 & -0.0635 & 3.6055 \end{pmatrix}, \quad \begin{pmatrix} 3.0101 & 0.1421 & -0.5671 \\ 0.1421 & 0.6660 & 0.0635 \\ -0.5671 & 0.0635 & 3.6055 \end{pmatrix},$$

for $P_{\{\emptyset\}} = P_{\{3\}} = P_{\{2,3\}}$, $P_{\{1\}} = P_{\{1,2\}} = P_{\{1,2,3\}}$, $P_{\{2\}}$ and $P_{\{1,3\}}$, respectively.

We now analyze an example with D not P-matrix. Let us consider the second order LCS with $A = \left(\begin{smallmatrix} -1 & -1 \\ 0 & -1 \end{smallmatrix}\right)$, $B = -A$, C identity matrix and D zero matrix. By phase plane analysis, one can verify that for any initial condition in the first quadrant, the state trajectory is unique and converges exponentially to zero. The cones $\mathcal{S}_\alpha$ and $\mathcal{S}_{\alpha_x}$ can be easily derived by modes enumeration. By applying Lemma 12 we obtained the following solution

$$P_{\{\emptyset\}} = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix},\ P_{\{1\}} = \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix},\ P_{\{2\}} = \begin{pmatrix} 1 & -3 \\ -3 & 4 \end{pmatrix}.$$

VII. Conclusion

The feasibility and solution sets of LCS have been represented by means of suitable cones. Then, sufficient exponential stability of the origin of the LCS have been proposed by formulating cone-copositive problems. The solution of corresponding sets of LMI lead to quadratic and piecewise quadratic Lyapunov functions able to prove the exponential stability of the origin in LCS. Examples demonstrate the effectiveness of the proposed approach.

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