Discrete-time Robust Hierarchical Linear-Quadratic Dynamic Games

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***Abstract*—In this paper, the theory of robust min-max control is extended to hierarchical and multi-player dynamic games for linear quadratic discrete time systems. The proposed game model consists of one leader and many followers, while the performance of all players is affected by disturbance. The Stackelberg-Nash-saddle equilibrium point of the game is derived and a necessary and sufficient condition for the existence and uniqueness of such a solution is obtained. In the infinite time horizon, it is shown that the solution of the Riccati equation is upper bounded under a condition which is called individual controllability. By assuming such a condition and using a time varying Lyapunov function the input-to-state stability of the hierarchical dynamic game is achieved, considering the optimal feedback strategies of the players and an arbitrary disturbance as the input.**

***Index Terms*—Dynamic Hierarchical game, Robust, Linear Quadratic, Stackelberg-Nash-Saddle point.**

I. Introduction

Hierarchical decision making has been applied widely in many engineering fields like smart grids, manufacturing and wireless communication networks [1]–[6]. The common aspect of those engineering applications is that at the higher level, a player decides first as a leader, while at the second level many players decide in reaction to the decision of the leader, as the followers. Therefore, there exists a competition between leader and followers and also among the followers, as well.

The problem of multi-player hierarchical decision making becomes more complicated when players decide repeatedly, in a dynamic environment. Dynamic games have been studied throughly in the literature of control theory [7]. The Nash and Stackelberg equilibrium points are the well-known solutions of a dynamic game, where the players decide simultaneously or as a leader-follower, respectively [8]–[11]. Some engineering applications of the dynamic games are given in [12], [13]. Recently, dynamic games have been applied to hierarchical decision making in fields like pricing and spectrum sharing in communication networks [14].

To address environmental uncertainties, the min-max control problem can be looked as a game theoretic perspective of robust H-inf control strategy in a dynamic environment [15]. In this framework, the problem is modeled as a zero-sum dynamic game, i.e., the uncertainty is considered as a player who wants to maximize the cost function while the decision maker wants to minimize it [15]. Some applications of dynamic min-max control can be found in security problems [16], [17]. Recently, the min-max control strategy has been also applied for control over network losses under TCP-like protocol, where the wireless network system is affected by link failures and packets drop [18], [19]. In [20] the min-max game is applied for decision making in a two player sequential game. In the continuous time domain the results of min-max game is extended to multi-player game and the conditions for existence and uniqueness of the Nash-worst case strategy of disturbance is given [21], [22]. However, to the best of the authors' knowledge, the robust decision making has not been explored in discrete time multi-player hierarchical dynamic games, yet. Recently, some research works have been carried out on Stackelberg and hierarchical games. In [23] a necessary and sufficient condition is given for the existence and uniqueness of the two-player Stackelberg game. An existence condition for equilibrium point of a multi-leaders and multi-followers static game is given in [24]. It is shown that the equilibrium point of such a hierarchical game is a solution of an optimization problem in which the solvability of the optimization problem is guaranteed under some mild conditions.

This paper addresses a formulation for dynamic discrete time hierarchical linear quadratic games, considering the uncertainties as an exogenous disturbance to the system. In the proposed dynamic game, the leader acts first and plays a Stackelberg game with the followers. In the second level, the followers play an n-player non-cooperative game and decide simultaneously, after the leader. All the players, including the leader and the followers, also play a min-max game with the disturbance. The Stackelberg-Nash-saddle point equilibrium of the game is derived that can be interpreted as the robust optimal strategy of the players. It is shown that the robust optimal closed loop strategies of the leader and the followers are in fact linear feedback strategies and the corresponding Riccati equations are given. A necessary and sufficient condition for the existence and uniqueness of the equilibrium point of the dynamic hierarchical game is also achieved. In addition, it is proved that under a certain controllability assumption which is called individual controllability the system under the proposed players' control strategies satisfies an input-to-state stability condition.

This paper is organized as follows: Section II describes the detailed system model and the problem formulation. Section III proposes the closed loop robust optimal strategies for the players. Conditions for the existence and uniqueness of the equilibrium points are also given in this section. The stability analysis of the system under the proposed feedback strategies is given in Sect. IV. Section V includes the simulation results and finally the paper is concluded in Section VI.

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II. System Model and Problem Formulations

The proposed hierarchical game consists of a set of $n+2$ players including a leader, n followers and a disturbance. The players decide in a dynamic environment with closed loop information about the state of the system in a finite horizon time-steps. The strategy of each player affects the objective function of all the players through the state space of the system. The disturbance is considered as a virtual player who wants to provide the worst case condition for the objective function of the players. In the lower level of the game, the followers play a dynamic Nash game, where the players decide simultaneously, without having the information of other players' strategies and by knowing the strategy of the leader. First, the best response of the followers to the other followers' strategies and disturbance is computed. In addition, for each follower, the best response (worst case) strategy of the disturbance is also computed as a function of each follower's strategy. Then, by intersecting the best responses of the followers and disturbance ($2n$ equations) the Nash-Saddle equilibrium point of the game is computed. The obtained strategy of the followers in this level will be used by the leader for decision making in the higher level, as the reaction function of the followers to the leader's strategy. Then the leader uses it to play a min-max game with disturbance. The resulted strategy in this level is a Stackelberg-Saddle point equilibrium strategy. Finally, the whole hierarchical game admits a Stackelberg-Nash-saddle equilibrium point.

Consider a dynamic system evolving according to the following difference equation

$$\begin{aligned} x_{k+1} =& Ax_k + B^{n+1}v_k + \sum_{i=1}^{n} B^i u_k^i + Dw_k \\ =& f(x_k, v_k, u_k^1, \dots, u_k^i, \dots, u_k^n, w_k), \end{aligned} \quad (1)$$

where $x_k \in \mathbb{R}^q$ is the state, $v_k \in \mathbb{R}^m$ is the leader's decision, $u_k^i \in \mathbb{R}^m$, $i = 1, 2, \dots, n$ is the i-th follower's decision and $w_k \in \mathbb{R}^p$ is the disturbance, and k points to the number of stage. The leader's decision is based on followers' rationality and both leader and followers are trying to set their control strategy in order to minimize their cost function over a finite number of stages, say N. Those cost functions are in quadratic form, as inspired by the robust game theoretic approach [15]. They can be written as

$$\begin{aligned} J_{0,N}^i(x_0, u_{0\bullet}^i, w_{0\bullet}) \triangleq& \sum_{k=0}^{N-1} \left(x_k^\top Q^i x_k + r^i \|u_k^i\|^2 - \gamma^2 \|w_k\|^2 \right) \\ &+ x_N^\top Q_N^i x_N, \quad i = 1, 2, \dots, n \end{aligned} \quad (2a)$$

$$\begin{aligned} J_{0,N}^{n+1}(x_0, v_{0\bullet}, w_{0\bullet}) \triangleq& \sum_{k=0}^{N-1} \left(x_k^\top Q^{n+1} x_k + r^{n+1} \|v_k\|^2 \right. \\ &\left. - \gamma^2 \|w_k\|^2 \right) + x_N^\top Q_N^{n+1} x_N, \end{aligned} \quad (2b)$$

where $Q^i, Q^{n+1} \in \mathbb{R}^{q \times q}$ and the terminal weights ($Q_N^i, Q_N^{n+1} \succcurlyeq 0$) are all nonnegative definite matrices (denoted by $\succcurlyeq 0$) and $\gamma > 0$, $r^i, r^{n+1} > 0$ are positive real scalars. Equations (2a), (2b) describe the followers' and the leader's cost functions, respectively, and weight the decision sequences $u_{0\bullet}^i$ (where $u_{0\bullet}^i \triangleq \{u_k^i\}_{k=0}^{N-1}$, analogously for other sequences) and $v_{0\bullet}$, together with the the state sequence $x_{0\bullet}$ (obtained from the initial state x_0 through the game dynamics (1)) and also the disturbance effects given by the sequence $w_{0\bullet}$. All of the players including the leader and followers play a min-max game against the disturbance w, here considered as a player who wants to maximize the cost function, while other players want to minimize it. γ is the attenuation parameter which is an estimation of the upper bound of the H-infinite norm of the transfer function from the disturbance to the output [15].

III. Equilibrium Analysis of the Hierarchical Dynamic Game

***Theorem** 1:* Consider the proposed hierarchical game including one leader and n-followers with objective functions defined in (2a), (2b) and the dynamic system (1) affected by the disturbance. If there exists a unique Stackelberg-Nash-saddle point equilibrium including the strategy of all the players, then the following conditions need to be satisfied

$$\Phi_k^i \triangleq \gamma^2 I_p - D^\top Z_{k+1}^i D \succ 0, \quad i = 1, 2, \dots, n+1 \quad (3)$$

where the symmetric nonnegative matrices $Z_k^i \succcurlyeq 0$ are the solutions of the following recursive difference Riccati equations, $0 \le k \le N$,

$$\begin{aligned} Z_k^i = Q^i + (H_k^i)^\top Z_{k+1}^i H_k^i + r^i {L_k^i}^\top L_k^i - \gamma^2 (L_k^{-i})^\top L_k^{-i} \\ i = 1, 2, \dots, n, n+1 \end{aligned} \quad (4)$$

with the terminal conditions

$$Z_N^i = Q_N^i \succcurlyeq 0, \; i = 1, 2, \dots, n, n+1 \quad (5)$$

and where

$$H_k^i = H_k^0 + D L_k^{-i}, \; i = 1, 2, \dots, n, n+1 \quad (6a)$$

$$H_k^0 = A'_k + B'_k L_k^{n+1} \quad (6b)$$

$$L_k^i = F_k^i (A + B^{n+1} L_k^{n+1}) \qquad i = 1, 2, \dots, n \quad (7)$$

$$L_k^{-i} = F_k^{-i} (A + B^{n+1} L_k^{n+1}) \qquad i = 1, 2, \dots, n \quad (8)$$

$$F_k = \sum_{i=1}^{n} B^i F_k^i \quad (9)$$

$$A'_k = A + F_k A, \;\; B'_k = B^{n+1} + F_k B^{n+1}, \quad (10)$$

such that the Stackelberg-Nash-Saddle point at each stage k will be linear in the state x_k through the gains L_k as below:

$$\left(u_k^{i\,*} \; w_k^{i\,*} \; v_k{}^* \; w_k^{n+1*} \right) = \left(L_k^i \; L_k^{-i} \; L_k^{n+1} \; L_k^{-(n+1)} \right) x_k, \quad (11)$$

where $i = 1, 2, \dots, n$ refer to the followers, $n+1$ points to the leader, and the indices $-1, -2, \dots, -n$ indicate the different disturbance's worst case feedback strategy for different followers and the index $i = -(n+1)$ refers to the disturbance's worst case feedback strategy for the leader.

Proof: Based on the bottom-up principle we want to solve the hierarchal game from down to top. Accordingly, for the sake of readability, the proof will be split into three different steps.

First step: followers' optimal strategies. In lower level, the sub-game perfect equilibrium point of the game is obtained by finding the Nash-saddle point equilibrium of the game among the followers and also between each follower and the disturbance. Consider the simultaneous decision making of the n-followers according to cost functions given in (2a). First, we want to find the optimal reaction functions of the followers to the strategy of the leader, by taking into account the state and system dynamics information (it is assumed they know it) and the worst case effect of disturbance on their own cost function. That means each follower could get different saddle point equilibria with respect to the disturbance. Thus we will denote w_k^i as the disturbance effect into the i-th follower's cost function. Assume that the system is in the k-th stage and all the cost-to-go functions in the next stages have been already optimized. The goal is to calculate the k-th stage saddle point equilibrium for all the followers. Then, by using the dynamic programming approach, we have the following value function recursion for the i-th player:

$$V_k^i(x) = \min_{u_k^i} \max_{w_k^i} \left\{ V_{k+1}^i(f(x, v_k, u_k^1, \dots, u_k^i, \dots, u_k^n, w_k^i)) + x^\top Q^i x + r^i \|u_k^i\|^2 - \gamma^2 \|w_k^i\|^2 \right\}, \;\; k < N \tag{12a}$$

$$V_N^i(x) = x^\top Q_N^i x \tag{12b}$$

where the corresponding cost-to-go function at the step k starting from the state x_k is

$$\begin{aligned} J_k^i(x_k, u_k^i, w_k^i) &\triangleq V_{k+1}^i(x_{k+1}) + x_k^\top Q^i x_k + r^i \|u_k^i\|^2 - \gamma^2 \|w_k^i\|^2 \\ &= x_{k+1}^\top Z_{k+1}^i x_{k+1} + x_k^\top Q^i x_k + r^i \|u_k^i\|^2 - \gamma^2 \|w_k^i\|^2 \\ &= (Ax_k + B^{n+1} v_k + \sum_{j=1}^n B^j u_k^j + D w_k^i)^\top Z_{k+1}^i \\ &\cdot (Ax_k + B^{n+1} v_k + \sum_{j=1}^n B^j u_k^j + D w_k^i) \\ &+ x_k^\top Q^i x_k + r^i \|u_k^i\|^2 - \gamma^2 \|w_k^i\|^2. \end{aligned} \tag{13}$$

In order to calculate the optimal strategies of the followers and the worst case strategy of disturbance (i.e., $\left(u_k^{i\,*}, w_k^{i\,*}\right)$), the first derivatives of the cost-to-go function with respect to u_k^i and w_k^i are set equal to zero as follows:

$$\begin{aligned} \frac{\partial J_k^i}{\partial u_k^i} = 0 \rightarrow (B^i)^\top Z_{k+1}^i \sum_{j=1}^n B^j u_k^j + r^i u_k^i + (B^i)^\top Z_{k+1}^i D w_k^i \\ = -(B^i)^\top Z_{k+1}^i (Ax_k + B^{n+1} v_k) \end{aligned} \tag{14a}$$

$$\begin{aligned} \frac{\partial J_k^i}{\partial w_k^i} = 0 \rightarrow D^\top Z_{k+1}^i \sum_{j=1}^n B^j u_k^j + (D^\top Z_{k+1}^i D - \gamma^2 I_p) w_k^i \\ = -D^\top Z_{k+1}^i (Ax_k + B^{n+1} v_k). \end{aligned} \tag{14b}$$

If we denote $\mathcal{B}_k \in \mathbb{R}^{mn \times mn}$, $\mathcal{D}_k \in \mathbb{R}^{pn \times pn}$ and $\mathcal{A}_k \in \mathbb{R}^{pn \times pn}$ as the blocks of the matrix α_k given in (15) such that

$$\alpha_k = \begin{pmatrix} \mathcal{B}_k & \operatorname{diag}(\mathcal{A}_k^\top) \\ \mathcal{A}_k & \mathcal{D}_k \end{pmatrix}, \tag{16}$$

then conditions (14) can be rewritten as:

$$\mathcal{B}_k \mathcal{U}_k^* + \operatorname{diag}(\mathcal{A}_k^\top) \mathcal{W}_k^* = \mathcal{E}_k (Ax_k + B^{n+1} v_k) \tag{17a}$$

$$\mathcal{A}_k \mathcal{U}_k^* + \mathcal{D}_k \mathcal{W}_k^* = \mathcal{F}_k (Ax_k + B^{n+1} v_k), \tag{17b}$$

with $\mathcal{E}_k = -[Z_{k+1}^1 B^1, \dots, Z_{k+1}^n B^n]^\top$, $\mathcal{F}_k = -[Z_{k+1}^1 D, \dots, Z_{k+1}^n D]^\top$ and $\mathcal{U}_k^*$ and $\mathcal{W}_k^*$ column vectors obtained by stacking the Nash-saddle-point values $u_k^{i\,*}$ and $w_k^{i\,*}$. As it is shown, there are matrix equations that should be solved simultaneously, in order to obtain the Nash-saddle-point equilibrium strategy of all followers in k-th stage of the game. System (17) has $(m+p)n$ linear equations with $(m+p)n$ unknowns. Since by hypothesis there exists a unique Stackelberg-Nash-saddle point equilibrium, then α_k is invertible (see Theorem 2 for details and conditions), and from (17) the reaction functions on the followers and the corresponding worst-case strategies disturbance will be

$$u_k^{i\,*} = F_k^i (Ax_k + B^{n+1} v_k) \qquad i = 1, 2, \dots, n \tag{18a}$$

$$w_k^{i\,*} = F_k^{-i} (Ax_k + B^{n+1} v_k), \qquad i = 1, 2, \dots, n, \tag{18b}$$

where

$$\left((F_k^1)^\top \cdots (F_k^n)^\top \quad (F_k^{-1})^\top \cdots (F_k^{-n})^\top\right)^\top = \alpha_k^{-1} \begin{pmatrix} \mathcal{E}_k \\ \mathcal{F}_k \end{pmatrix}. \tag{19}$$

Note that in (19) the matrices α_k and $\mathcal{E}_k$ and $\mathcal{F}_k$ depends only on system matrices and Z_{k+1}^i, i.e., the Riccati solution at the stage $k+1$ which are all available at sage k. The value $w_k^{i\,*}$ corresponds to the worst case strategy that could be caused by disturbance on the i-th follower's cost function. In fact, there is just one disturbance as an arbitrary input to the dynamic equation (1), but the expectation of each player from the "worst case" effect of such an unknown disturbance on the objective function is different and it is obtained by maximizing the objective function of that player over possible disturbance strategies as an input. This maximization gives different worst case strategies for different objective functions which are called worst case strategies of disturbance for that player. It is clear from (18) that the optimal strategies of the followers are dependent on the strategy of the leader. Once the leader makes its decision, these strategies are evaluated.

Second step: leaders' optimal strategy. Let us focus now on calculating the Stackelberg-saddle point equilibrium of the game in higher level which is a Stackelberg game between leader and followers and also a min-max game between leader and disturbance. By considering that the leader knows the reaction functions of followers and uses those functions in its optimal decision making, the followers' reaction functions are put into the states equation and then the state space equation is changed to the following form:

$$\begin{aligned} x_{k+1} &= Ax_k + B^{n+1} v_k + \sum_{i=1}^n \left(B^i F_k^i (Ax_k + B^{n+1} v_k)\right) + D w_k \\ &= (A + F_k A) x_k + (B^{n+1} + F_k B^{n+1}) v_k + D w_k \\ &= A_k' x_k + B_k' v_k + D w_k. \end{aligned} \tag{20}$$

Note that the optimal value (worst case strategy) of disturbance which was calculated in (18b) is not replaced in (20). The reason is that we also want to calculate the worst case condition for the leader. Again, by applying the dynamic programming approach, the leader's cost-to-go function at the

$$
\alpha_k = \left(\begin{array}{cccc|cccc}
(B^1)^\top Z^1_{k+1}B^1 + r^1 I_m & (B^1)^\top Z^1_{k+1}B^2 & \cdots & (B^1)^\top Z^1_{k+1}B^n & (B^1)^\top Z^1_{k+1}D & 0 & \cdots & 0 \\
(B^2)^\top Z^2_{k+1}B^1 & (B^2)^\top Z^2_{k+1}B^2 + r^2 I_m & \cdots & (B^2)^\top Z^2_{k+1}B^n & 0 & (B^2)^\top Z^2_{k+1}D & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(B^n)^\top Z^n_{k+1}B^1 & (B^n)^\top Z^n_{k+1}B^2 & \cdots & (B^n)^\top Z^n_{k+1}B^n + r^n I_m & 0 & 0 & \cdots & (B^n)^\top Z^n_{k+1}D \\
\hline
D^\top Z^1_{k+1}B^1 & D^\top Z^1_{k+1}B^2 & \cdots & D^\top Z^1_{k+1}B^n & -\Phi^1_k & 0 & \cdots & 0 \\
D^\top Z^2_{k+1}B^1 & D^\top Z^2_{k+1}B^2 & \cdots & D^\top Z^2_{k+1}B^n & 0 & -\Phi^2_k & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
D^\top Z^n_{k+1}B^1 & D^\top Z^n_{k+1}B^2 & \cdots & D^\top Z^n_{k+1}B^n & 0 & 0 & \cdots & -\Phi^n_k
\end{array}\right) \tag{15}
$$

k-th stage is

$$
\begin{aligned}
&J^{n+1}_k(x_k, v_k, w^{n+1}_k) \triangleq x^\top_{k+1} Z^{n+1}_{k+1} x_{k+1} + x^\top_k Q^{n+1} x_k \\
&+ r^{n+1}\|v_k\|^2 - \gamma^2 \|w^{n+1}_k\|^2 = \\
&(A'_k x_k + B'_k v_k + D w^{n+1}_k)^\top Z^{n+1}_{k+1} (A'_k x_k + B'_k v_k + D w^{n+1}_k) \\
&+ x^\top_k Q^{n+1} x_k + r^{n+1}\|v_k\|^2 - \gamma^2 \|w^{n+1}_k\|^2.
\end{aligned} \tag{21}
$$

Taking the derivatives with respect to the control strategy and disturbance and putting them equal to zero we have

$$
\begin{aligned}
\frac{\partial J^{n+1}_k}{\partial v_k} =& 0 \rightarrow (B'_k)^\top Z^{n+1}_{k+1} B'_k v_k + r^{n+1} v_k \\
&+ (B'_k)^\top Z^{n+1}_{k+1} D w^{n+1}_k = -(B'_k)^\top Z^{n+1}_{k+1} A'_k x_k
\end{aligned} \tag{22a}
$$

$$
\begin{aligned}
\frac{\partial J^{n+1}_k}{\partial w^{n+1}_k} &= 0 \rightarrow D^\top Z^{n+1}_{k+1} B'_k v_k + (D^\top Z^{n+1}_{k+1} D - \gamma^2) w^{n+1}_k \\
&= -D^\top Z^{n+1}_{k+1} A'_k x_k.
\end{aligned} \tag{22b}
$$

Considering the uniqueness of the Stackelberg-saddle equilibrium by hypothesis, equation (23) gives the optimal strategies of the leader and corresponding worst-case strategy of disturbance:

$$
\begin{pmatrix} v_k{}^* & w^{n+1*}_k \end{pmatrix} = \begin{pmatrix} L^{n+1}_k & L^{-(n+1)}_k \end{pmatrix} x_k, \tag{23}
$$

where

$$
\begin{aligned}
\begin{pmatrix} L^{n+1}_k \\ L^{-(n+1)}_k \end{pmatrix} &= \begin{pmatrix} r^{n+1} I + (B'_k)^\top Z^{n+1}_{k+1} B'_k & (B'_k)^\top Z^{n+1}_{k+1} D \\ -D^\top Z^{n+1}_{k+1} B'_k & \gamma^2 I - D^\top Z^{n+1}_{k+1} D \end{pmatrix}^{-1} \\
&\times \begin{pmatrix} -(B'_k)^\top Z^{n+1}_{k+1} A'_k \\ D^\top Z^{n+1}_{k+1} A'_k \end{pmatrix}.
\end{aligned} \tag{24}
$$

Once the robust optimal strategy of the leader is derived then the corresponding strategies of the followers in (18) is evaluated. Replacing (23) into (18), the expressions in (7), (8) and (11) are proven. One can verify the *necessary conditions for the existence* of the Stackelberg-Nash-saddle point equilibrium of the game: it comes out that the cost functions must be convex-concave, with respect to the players (follower or leader) and the disturbance in every stage, respectively:

$$
\frac{\partial^2 J^i_k}{\partial {u^i_k}^2} = (B^i)^\top Z^i_{k+1} B^i + r^i I_m \succ 0 \tag{25a}
$$

$$
\frac{\partial^2 J^i_k}{\partial {w^i_k}^2} = D^\top Z^i_{k+1} D - \gamma^2 I_p = -\Phi^i_k \prec 0 \tag{25b}
$$

$$
i = 1, 2, \ldots, n, n+1.
$$

Thus, conditions (3) should be applied as necessary conditions. Note that conditions (25a) are always guaranteed by the nonnegative definiteness of Z^i_k, since $r^i > 0$.

Third step: Riccati equations. Now, in order to complete the proof, we need to derive the Riccati equations and show the nonnegative definiteness of their solutions. To this aim, the obtained control strategies are replaced into the state equation and then into the cost-to-go functions, as well. Then the value function of the leader and followers can be written as

$$
\begin{aligned}
V^i_k(x_k) =& x^\top_k (A'_k + B'_k L^i_k + D L^{-(i)}_k)^\top Z^i_{k+1} \\
&\times (A'_k + B'_k L^i_k + D L^{-(i)}_k) x_k \\
&+ x^\top_k Q^i x_k + r^i \|L^i_k x_k\|^2 - \gamma^2 \|L^{-(i)}_k x_k\|^2 \\
=& x^\top_k ((H^i_k)^\top Z^i_{k+1} H^i_k + r^i (L^i_k)^\top L^i_k \\
&- \gamma^2 (L^{-(i)}_k)^\top L^{-(i)}_k + Q^i) x_k = x^\top_k Z^i_k x_k.
\end{aligned} \tag{26}
$$

for $i = 1, 2, \ldots, n, n+1$. Finally we have to prove the nonnegative definiteness of the Riccati equations solutions. From (14b) and (22b), it comes out

$$
\Phi^i_k L^{-i}_k = D^\top Z^i_{k+1} (A'_k + B'_k L^i_k) = D^\top Z^i_{k+1} H^0_k. \tag{27}
$$

We will rewrite equation (26), by using (6a) and (27):

$$
\begin{aligned}
x^\top_k Z^i_k x_k =& x^\top_k \left(Q^i + (H^0_k)^\top Z^i_{k+1} H^0_k + (H^0_k)^\top Z^i_{k+1} D L^{-i}_k \right. \\
&+ (D L^{-i}_k)^\top Z^i_{k+1} H^0_k + (L^{-i}_k)^\top D^\top Z^i_{k+1} D L^{-i}_k \\
&\left. + r^i (L^i_k)^\top L^i_k - \gamma^2 (L^{-i}_k)^\top L^{-i}_k \right) x_k \\
=& x^\top_k \left(Q^i + (H^0_k)^\top Z^i_{k+1} H^0_k + r^i (L^i_k)^\top L^i_k \right. \\
&\left. + (L^{-i}_k)^\top \Phi^i_k L^{-i}_k \right) x_k
\end{aligned} \tag{28}
$$

Clearly, conditions (3) together with $r^i > 0$ and $Q^i \succcurlyeq 0$, guarantee the nonnegative definiteness of Z^i_k as given in (28) for all $i = 1, 2, \ldots, n+1$, provided $Z^i_{k+1} \succcurlyeq 0$. Since $Z^i_N =$

Algorithm 1 Computation of the equilibrium point

1: Initialize the Riccati matrices in (5) and set $k+1 := N$.
2: Compute the gains F^i_k and F^{-i}_k using (19).
3: Compute L^{n+1}_k and $L^{-(n+1)}_k$ from (24).
4: Compute L^i_k and L^{-i}_k from (7) and (8).
5: Update the Riccati matrices in (4).
6: **If** $k > 1$,
7: set $k := k-1$ and go to Step 2.
8: **else**
9: end.

$Q_N^i \succcurlyeq 0$ that is proven by induction. ■

According to the results of Theorem 1, Algorithm 1 gives the computational procedure of the optimal gains and Riccati matrices.

***Theorem** 2:* If conditions (3) of Theorem 1 are satisfied then the cost functions are strictly convex-concave and there exists a unique Stackelberg-Nash-saddle point equilibrium including the strategy of all players if and only if $I + \sum_{i=1}^{n} B^i(r^i)^{-1}(B^i)^\top Z_{k+1}^i[I + D(\Phi_k^i)^{-1}D^\top Z_{k+1}^i]$ is nonsingular.

Proof: From relations (25) it is clear that positive definiteness of matrices (3) are equivalent to say that the cost functions are strictly convex-concave. Thus the existence and uniqueness of the Stackelberg-Nash-saddle point equilibrium are equivalent. Considering the conditions related to the optimal strategies of followers in (17), if the conditions (3) are satisfied in a strict sense, then $\mathcal{D}_k$ is invertible and by a Schur complement argument we get

$$\mathcal{W}_k^* = \mathcal{D}_k^{-1}\left(-\mathcal{A}_k\mathcal{U}_k^* + \mathcal{F}_k(Ax_k + B^{n+1}v_k)\right), \tag{29a}$$

$$\mathcal{U}_k^* = \left(\mathcal{B}_k - \text{diag}(\mathcal{A}_k^\top)\mathcal{D}_k^{-1}\mathcal{A}_k\right)^{-1} \times \left(\mathcal{E}_k - \text{diag}(\mathcal{A}_k^\top)\mathcal{D}_k^{-1}\mathcal{F}_k\right)(Ax_k + B^{n+1}v_k). \tag{29b}$$

It is not difficult to show that the matrix $\mathcal{M}_k = \mathcal{B}_k - \text{diag}(\mathcal{A}_k^\top)\mathcal{D}_k^{-1}\mathcal{A}_k$ can be partitioned into the following blocks:

$$(\mathcal{M}_k)_{ij} = (B^i)^\top Z_{k+1}^i D(\Phi_k^i)^{-1}D^\top Z_{k+1}^i B^j + (B^i)^\top Z_{k+1}^i B^j, \quad i,j = 1,\ldots,n, \ i \neq j \tag{30a}$$

$$(\mathcal{M}_k)_{ii} = (B^i)^\top Z_{k+1}^i D(\Phi_k^i)^{-1}D^\top Z_{k+1}^i B^i + (B^i)^\top Z_{k+1}^i B^i + r^i I_m, \quad i = 1,\ldots,n. \tag{30b}$$

We can write

$$\mathcal{M}_k = R + \begin{pmatrix} (B^1)^\top M_k^1 \\ (B^2)^\top M_k^2 \\ \vdots \\ (B^n)^\top M_k^n \end{pmatrix} \underbrace{\begin{pmatrix} B^1 & B^2 & \cdots B^n \end{pmatrix}}_{\tilde{B}} = R + \tilde{P}_k\tilde{B}, \tag{31}$$

where $R = \text{diag}\,(r^i I_m)$ is a block diagonal matrix and

$$M_k^i = Z_{k+1}^i[I_q + D(\Phi_k^i)^{-1}D^\top Z_{k+1}^i] \succcurlyeq 0. \tag{32}$$

By applying the matrix inversion lemma it comes out that

$$\mathcal{M}_k^{-1} = R^{-1} - R^{-1}\tilde{P}_k S_k^{-1}\tilde{B}R^{-1}, \tag{33}$$

where

$$S_k = I_q + \tilde{B}R^{-1}\tilde{P}_k = I_q + \sum_{i=1}^{n} B^i(r^i)^{-1}(B^i)^\top M_k^i. \tag{34}$$

Therefore, nonsingularity of S_k implies the nonsingularity of $\mathcal{M}_k$ and the existence and uniqueness of the Nash-saddle equilibrium point. The existence and uniqueness of the Stackelberg-saddle equilibrium point holds thanks to the invertibility of the matrix in (24), implied by (3) and $r^{n+1} > 0$ (a Schur complement argument easily shows that). ■

***Remark** 1:* Note that here the increasing property of Z_k^i is not necessarily guaranteed like conventional LQ problem [25] or min-max H-inf problem [15] since in our case the matrix H_k^i is time varying which is not the case of those problems. We will show this property via an example in the simulation results.

***Lemma** 1:* If $Q_N^i \succ 0$, $i = 1, 2, \ldots, n+1$ and the matrices $\begin{pmatrix} C^{i\top} & A^\top \end{pmatrix}^\top$, with $Q^i = C^{i\top}C^i$, are full rank for all $i = 1, 2, \ldots, n+1$, then the solutions of the Riccati equations (4)-(5) are positive definite.

Proof: Let's first show that $Z_{k+1}^i \succ 0$ implies $Z_k^i \succ 0$. The proof follows by contradiction. Assume $\exists\, x_k \neq 0 \,|\, x_k^\top Z_k^i x_k = 0$ (Z_k^i is nonnegative definite): by using (28), it necessarily follows that all those terms (that are nonnegative definite) have to be equal to zero. Moreover, since $Z_{k+1}^i \succ 0$ and $\Phi_k^i \succ 0$, then

$$C^i x_k = H_k^0 x_k = L_k^i x_k = L_k^{-i} x_k = 0, \quad i = 1, 2, \ldots, n+1. \tag{35}$$

According to (27), $L_k^{-i}x_k = 0$ is satisfied by $H_k^0 x_k = 0$. By looking at (7), we have

$$L_k^i x_k = F_k^i A x_k + F_k^i B^{n+1} L_k^{n+1} x_k = 0 \tag{36}$$

and, thus, $F_k^i A x_k = 0$. From (6b) and taking into account the fact that $L_k^{n+1}x_k = 0$ and $F_k^i A x_k = 0$, it follows

$$H_k^0 x_k = A_k' x_k = A x_k = 0. \tag{37}$$

Then the existence of $x_k \neq 0 \,|\, x_k^\top Z_k^i x_k = 0$, implies

$$\begin{pmatrix} C^i \\ A \end{pmatrix} x_k = 0, \tag{38}$$

which contradicts the hypothesis and it is concluded that $Z_k^i \succ 0$. Since $Z_N^i = Q_N^i \succ 0$, by induction we get $Z_k^i \succ 0, \ \forall k \leq N, \ i = 1, 2, \ldots, n+1$. ■

Of course, since the Nash-Stackelberg-saddle point strategy is a linear feedback applied to a linear system, the cost functions, and thus the solutions of the Riccati equations, are upper bounded for any finite time interval. We will now show that Riccati equations solutions have an upper bound also for an infinite time interval.

***Lemma** 2:* If the pairs (A_k^i, B^i), $i = 1, 2, ..., n+1$ are controllable, then the total cost functions $J_{0,N}^i$ of the leader and followers (equations (2)) are upper bounded for any $0 < N \leq +\infty$ under the Nash-Stackelberg-saddle point strategy of disturbance and followers and leader's control strategies given by (11), where

$$A_k^i = A + DL_k^{-i} + B^{(n+1)}L_k^{n+1} + \sum_{j \neq i} B^j L_k^j \tag{39a}$$

$$A_k^{n+1} = A + DL_k^{-(n+1)} + \sum_j B^j L_k^j. \tag{39b}$$

Furthermore, the corresponding solutions of the Riccati equations, denoted by $Z_{k,N}^i$ (for highlighting the fact that N is not fixed a priori), are also bounded from above, i.e., there exists a symmetric matrix $\bar{Z}$ such that $\bar{Z} \succcurlyeq Z_{k,N}^i \ \forall N \geq k \geq 0$.

Proof: Let us consider the controllability of one pair (A_k^i, B^i) (individual controllability) of the following system:

$$x_{k+1} = A_k^i x_k + B^i u_k^i. \tag{40}$$

It follows that there exists a set of control strategies $\tilde{U}_i^{t_i} = \{\tilde{u}_k^i\}_{k=0}^{t_i}$ for the dynamic system (40) so that $\tilde{U}_i$ steers any initial state x_0 to zero in a finite number (let's say t_i) of steps. Let us extend such control strategy over the whole number of stages $k \in [0,\, N-1]$ by adding all zeros to the control sequence, such that $\tilde{U}_i \triangleq \{\tilde{u}_k^i\}_{k=0}^{t_i} \cup \{0, 0, \dots, 0\}$. By considering linear feedback strategies of the other players, chosen as the optimal gains (11), and the worst case strategy of the disturbance from the i-th player's perspective, it is possible to apply the control strategy $\tilde{U}_i$ into the system (40) in such a way that the cost-to-go function J_k^i will be equal to zero for $k \geq t_i$ and therefore, the total cost of the i-th player will be bounded, under such a control strategy, for any $N > t_i$.

It is straightforward to verify that the optimal control strategy of the i-th player, $\{u_k^{i*}\}_{k=0}^{N-1}$, given by the Nash-Stackelberg-saddle point (11), corresponds to the optimal control strategy for the dynamic system (40) that minimizes the following cost function

$$J_{0,N}^i(x_0, u_{0\bullet}^i, L_{0\bullet}^{-i} x_{0\bullet}). \tag{41}$$

Since the set of control inputs $\{u_k^{i*}\}_{k=0}^{N-1}$ minimizes the total cost of the i-th player, given the equilibrium strategy of the other players and also disturbance, we have the following inequalities

$$\begin{aligned} 0 &\leq J_{0,N}^i(x_0, u_{0\bullet}^*, L_{0\bullet}^{-i} x_{0\bullet}{}^*) \leq J_{0,N}^i(x_0, \tilde{U}_i, L_{0\bullet}^{-i} x_{0\bullet}) \\ &\leq x_0^\top \bar{Z}^i x_0, \end{aligned} \tag{42}$$

where the first inequality is due to the nonnegative definiteness of the Riccati equation solutions while the latter one is related to the fact that $J_{0,N}^i(x_0, \tilde{U}_i, L_{0\bullet}^{-i} x_{0\bullet})$ is a quadratic form depending only on x_0 and $\tilde{U}_i$ is a linear function of x_0 [26] and it is equal to zero for $k \geq t_i$. Thus it comes out

$$0 \preccurlyeq Z_{0,N}^i \preccurlyeq \bar{Z}^i \quad i = 1, 2, \dots, n+1,\ \forall N \geq 0. \tag{43}$$

■

***Lemma** 3:* If the pairs $\left(A_k^i, B^i\right)$, $i = 1, 2, \dots, n+1$ in (39) are controllable and the pairs $\left(A, C^i\right)$, $i = 1, 2, \dots, n+1$ are observable, then there exists an upper bound for Z_k^i, denoted by $\overline{Z}^i$, and also a lower bound denoted by $\underline{Z}^i$ such that $0 \prec \underline{Z}^i \preccurlyeq Z_k^i \preccurlyeq \overline{Z}^i$ $\forall k\, i = 1, 2, \dots, n+1$.

Proof: The proof easily follows by previous lemmas and by considering that the observability of the pairs $\left(A, C^i\right)$ implies the full rank condition on the matrix $\begin{pmatrix} {C^i}^\top & A^\top \end{pmatrix}^\top$. Of course $\overline{Z}^i$ can be chosen as $\lambda_{\max}(\bar{Z}^i) I$ and $\underline{Z}^i = \min_k\left(\lambda_{\min}(Z_k^i)\right) I$. ■

IV. Stability Analysis

This section aims to investigate the stability of the closed loop system, considering the optimal feedback control signal of the leader and followers in presence of an arbitrary disturbance. It should be mentioned that the worst case feedback of the disturbance is not "implemented" (by disturbance) in reality but it is used just for computing the players' control strategy that is robust against an unknown disturbance. For that reason in this analysis the disturbance is considered as an arbitrary bounded input to the system and the input-to-state stability of the system (1) is studied.

***Theorem** 3:* Assuming the conditions in Theorem 1, if there exists a player j such that the pair (A_k^j, B^j) is controllable and the pair (A, C^j) is observable where, $Q^j = {C^j}^\top C^j$, then the system (1), with the closed loop state feedback $u_k^i = L_k^i x_k,\ i = 1, 2 \dots, n$ and $v_k = L_k^{n+1} x_k$, is input-to-state stable with respect to the disturbance w.

Proof: To prove the input-to-state stability we just need to prove that the system is globally asymptotically stable when $w = 0$ [27], i.e., when the closed loop dynamics are

$$x_{k+1} = A x_k + \sum_{i=1}^{n+1} B^i L_k^i x_k = \left(A_k' + B_k' L_k^{n+1}\right) x_k. \tag{44}$$

For this purpose, the following time-varying Lyapunov function is used:

$$V(x, k) = x^\top Z_k^j x. \tag{45}$$

By Lemma 3 we have

$$0 \prec \underline{Z}^j \preccurlyeq Z_k^j \preccurlyeq \overline{Z}^j. \tag{46}$$

If we consider $\eta = \lambda_{\min}(\underline{Z}^j)$ and $\rho = \lambda_{\max}(\overline{Z}^j)$ the proposed Lyapunov function lays in the following bounds:

$$\eta \|x\|^2 \leq V(x, k) \leq \rho \|x\|^2. \tag{47}$$

Then, the difference of the Lyapunov function, defined as

$$\Delta V(x, k) \triangleq V(x^{k+}, k+1) - V(x, k), \tag{48}$$

becomes (note that $x^{k+} \triangleq \left(A_k' + B_k' L_k^{n+1}\right) x$),

$$\begin{aligned} \Delta V(x, k) &= ((A_k' + B_k' L_k^{n+1}) x)^\top \left(Z_{k+1}^j\right) \left(A_k' + B_k' L_k^{n+1}) x\right) \\ &- x^\top \left(Q^j + (H_k^j)^T Z_{k+1}^j H_k^j + r^j (L_k^j)^T L_k^j - \gamma^2 (L_k^{-j})^T L_k^{-j}\right) x. \end{aligned} \tag{49}$$

After some simplifications we have

$$\begin{aligned} \Delta V(x, k) = &- x^\top \left(Q^j + 2(A_k' + B_k' L_k^{n+1})^\top Z_{k+1}^j D L_k^{-j}\right) x \\ &+ x^\top \left(r^j {L_k^j}^\top L_k^j - \gamma^2 (L_k^{-j})^\top L_k^{-j}\right) x \\ &+ x^\top (L_k^{-j})^\top (D^\top Z_{k+1}^j D) L_k^{-j} x. \end{aligned} \tag{50}$$

From (27), it is obtained that

$$x^\top (L_k^{-j})^\top \Phi_k^j L_k^{-j} x = x^\top (A'_k + B'_k L_k^{n+1})^\top Z_{k+1}^j D L_k^{-j} x. \tag{51}$$

By replacing (51) into (50)

$$\Delta V(x, k) = -x^\top \left(\ Q^j + r^j {L_k^j}^\top L_k^j + (L_k^{-j})^T \Phi_k^j L_k^{-j}\ \right) x_k. \tag{52}$$

Hence, we have

$$\Delta V(x, k) \leq 0. \tag{53}$$

In the case of equality we obtain

$$x^\top Q^j x = 0,\ L_k^j x = 0, \tag{54}$$

and thus the state of the system will asymptotically moves towards the region satisfying

$$C^j x = L_k^j x = 0. \quad (55)$$

From (44) in such region it holds

$$x_{k+1} = Ax_k, \ C^j x_k = 0, \quad (56)$$

but, since (A, C^j) is observable, it is not possible to have $C^j x = 0$ unless $x = 0$ and as a result the system is input-to-state stable. ■

V. SIMULATION

In this section, an illustrative example of proposed hierarchical dynamic system is presented. The number of state variables and the followers are considered equal to two. The states space model, initial conditions and output of the system are given as follows:

$$x_0 = \begin{pmatrix} 5 & 3 \end{pmatrix}^\top$$
$$x_{k+1} = \begin{pmatrix} 5 & 1 \\ 8 & 4 \end{pmatrix} x_k + \begin{pmatrix} 3 \\ 4 \end{pmatrix} v_k + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u_k^1 + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u_k^2 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} w_k. \quad (57)$$

The leader's control coefficient is set bigger than the followers, since the leader's decision is more effective in some practical cases. The disturbance is considered as a uniformly distributed noise, bounded in $[-0.5, 0.5]$. The value of other parameters are given in Table I. Note that the disturbance is considered as its actual value in the state equation (57). Nevertheless, since the players do not have access to the disturbance bound, they apply the proposed min-max scheme to act robustly against disturbance. The optimal control strategies of the players is shown in Fig. 1. As we can see, because of the effect of actual disturbance, although the system is disturbance-to-state stable (see Theorem 3), the control strategies are still affected by a zero mean bounded disturbance. The worst case strategies of disturbance and also the actual disturbance applied to the state equation are shown in Fig. 2. The optimal cost of the players considering the worst case strategy of disturbance is shown in Fig. 3. As we can see, the cost of the players in Fig. 3 is strictly increasing but that is not true for the Riccati matrices of the players. We mentioned this point in the paper in Remark 1. We demonstrate this property by plotting the first principal minor of the Riccati matrices in Fig. 4.

TABLE I
SYSTEM PARAMETERS

Parameter	Leader	First Follower	Second Follower
r^i	0.3	1	2
Q_N^i	$0.91 \cdot I_2$	$0.56 \cdot I_2$	$0.8 \cdot I_2$
Q^i	$5 \cdot I_2$	$2 \cdot I_2$	$4 \cdot I_2$

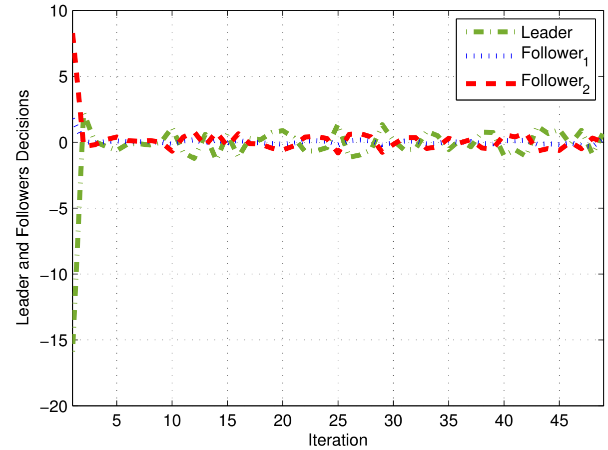

Fig. 1. Players' decisions

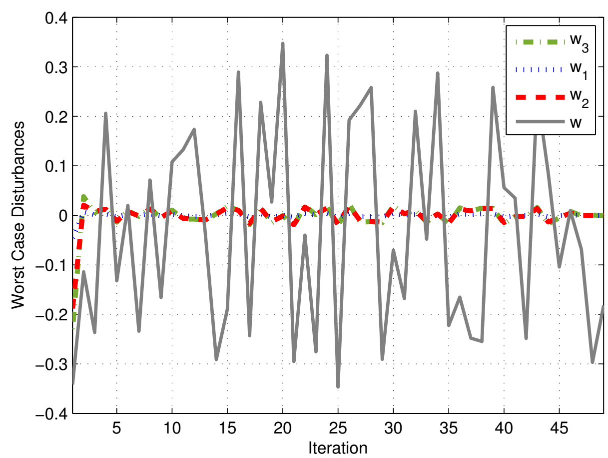

Fig. 2. Players' cost function with the actual disturbances

VI. CONCLUSION

In this paper the H-inf min-max problem was extended to a hierarchical multi-player robust game, by considering a discrete time linear state space model and quadratic cost functions of the players in presence of a disturbance. In the lower level of the hierarchical game the sub-game Nash-saddle equilibrium point was derived by intersecting the followers' best responses strategies together with disturbance's worst case strategies. In the higher level, where the leader plays a Stackelberg game with followers and also participates in a min-max game with the disturbance, the Stackelberg-Nash-saddle equilibrium point of the game was computed.

It has been showed how the considered game does not satisfy usual properties of the conventional LQ or min-max H-inf problems, e.g., the monotonicity of the Riccati equation. Nevertheless, conditions for the existence and uniqueness of the equilibrium point of the game was given and, moreover, by introducing the individual controllability assumption, it was proven the input-to-state stability of the infinite time horizon game played with optimal time varying strategies at the Stackelberg-Nash-saddle equilibrium point. The given results

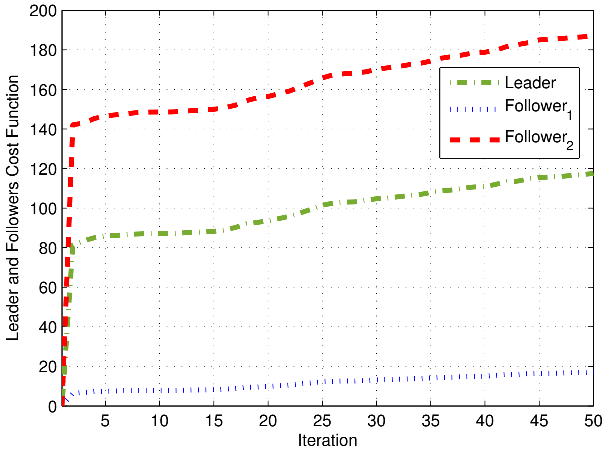

Fig. 3. Players' cost function with the worst-case strategies of disturbances

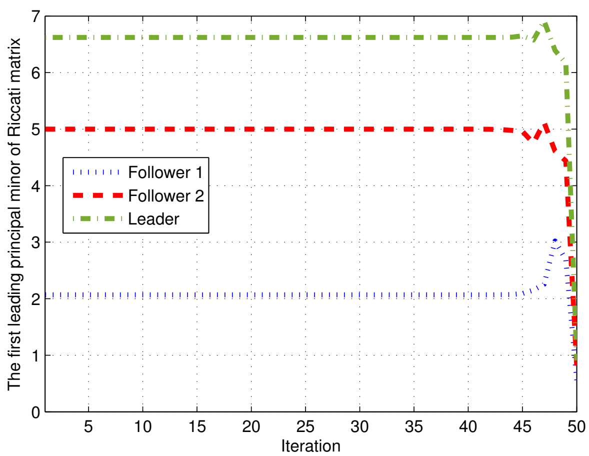

Fig. 4. The first leading principal of Riccati matrix of the players

open the way to further investigation of a such kind of game, in particular by considering problems like the convergence of the Riccati equations, the existence of the corresponding algebraic equations solution, the look for simple conditions that guarantee the individual controllability.

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