Stability Analysis of Conewise Linear Systems with Sliding Modes

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***Abstract*— In this paper cone-copositive piecewise quadratic Lyapunov functions (PWQ-LFs) for the stability analysis of conewise linear systems with the possible presence of sliding modes are proposed. The existence of a PWQ-LF is formulated as the feasibility of a cone-copositive programming problem which is represented in terms of linear matrix inequalities with equality constraints. An algorithm for the construction of a PWQ-LF is provided. Examples show the effectiveness of the approach in the presence of stable and unstable sliding modes.**

I. Introduction

Piecewise linear systems represent a particular class of hybrid systems characterized by a partition of the state space into regions where system dynamics can be described by linear models [1]. We consider piecewise linear systems where the state space partition consists of convex polyhedral cones. In the interior of each cone the dynamics are linear time-invariant, while on the cones boundaries the system dynamics are described by the convex hull of a finite set of known matrices, thus allowing the existence of sliding modes [2]. Such systems are called conewise linear systems [3], and can be viewed as a particularization of piecewise linear differential inclusions [4], state-dependent switched linear systems [5], or linear parameter varying systems [6]. Despite their apparent modeling simplicity, the stability analysis of conewise linear systems is a hard issue, the more so in the presence of sliding behavior. The simplest way to tackle the problem consists in employing a common Lyapunov function [7]. To reduce conservativeness one could use the multiple Lyapunov functions approach, i.e., to combine Lyapunov functions defined over different regions of the state space, see [5], [8]. In particular, when the regions of the state space are convex polyhedra, piecewise quadratic Lyapunov functions (PWQ-LFs) can be employed for solving the stability problem in terms of linear matrix inequalities (LMIs). The LMIs are typically obtained by applying the $\mathcal{S}$-procedure with quadratic forms representing regions that include the polyhedra [9]–[11]. The $\mathcal{S}$-procedure is used also in [12] where sliding modes are considered. The stability analysis for piecewise linear systems with sliding modes is not a trivial issue, indeed. That has been remarkably investigated also in [4], [7], [13] and the Introduction in [12] clearly reports their limitations.

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In this paper the results presented by the authors in [14], [15] and applied to Lur'e systems in [16] are extended by including the possible existence of sliding modes and by proposing an iterative procedure for the construction of a PWQ-LF. The technique explicitly considers the conic constraints by formulating a so called cone-copositive problem [17], without using the $\mathcal{S}$-procedure and outer approximations of the cones. The cone-copositive problem together with the sliding conditions are then reformulated in terms of LMIs. Sufficient conditions for the existence of a continuous PWQ-LF in terms of these LMIs with continuity equality constraints are derived, thus guaranteeing the global exponential stability of the origin of the conewise linear system. An algorithm for the construction of the PWQ-LF is also given. The effectiveness of the proposed approach is illustrated by analyzing several examples.

II. Preliminaries

In this section it is shown how cone-copositive problems can be solved by translating them into equivalent copositive problems and then expressed in terms of LMIs. To this aim some preliminary definitions are recalled.

Given λ points $\{v_\ell\}_{\ell=1}^{\lambda}$ with $v_\ell \in \mathbb{R}^n$ and $\ell \in \mathbb{N}$, the conic hull is the set of points $v \in \mathbb{R}^n$ such that $v = \sum_{\ell=1}^{\lambda} \theta_\ell v_\ell$, with $\theta_\ell \in \mathbb{R}_+$. A convex hull $\overline{\text{co}}\left\{\{v_\ell\}_{\ell=1}^{\lambda}\right\}$ is the conic hull with $\sum_{\ell=1}^{\lambda} \theta_\ell = 1$. When the points $\{v_\ell\}_{\ell=1}^{\lambda}$ are affinely independent, i.e., the $\lambda - 1$ points $v_2 - v_1, \ldots, v_\lambda - v_1$ are linearly independent, the convex hull $\overline{\text{co}}\left\{\{v_\ell\}_{\ell=1}^{\lambda}\right\}$ is a $(\lambda-1)$-simplex, and $\{v_\ell\}_{\ell=1}^{\lambda}$ are called *vertices* of the simplex. Clearly in order to define a simplex in $\mathbb{R}^n$ we need $\lambda \leq n+1$. A set $\mathcal{C} \subset \mathbb{R}^n$ is a simplicial cone if it is the conic hull of n linearly independent points. Polyhedral cones with nonempty interiors (proper polyhedral cones) can be always partitioned into a finite number of simplicial cones [18]. Hereinafter, without loss of generality, simplicial cones are considered. Given a simplicial cone $\mathcal{C} \subset \mathbb{R}^n$ there exists a nonsingular matrix $R \in \mathbb{R}^{n\times n}$, such that for any $v \in \mathcal{C}$ one can write $v = R\theta$ where $\theta \in \mathbb{R}_+^n$. The matrix R identifies the so-called *$\mathcal{V}$-representation* of the cone and its columns are the *extremal rays* of the cone. Each extremal ray is uniquely defined up to a positive multiple.

Given a set $X \subseteq \mathbb{R}^n$ and a finite positive integer η, a *partition* of X is the family $\mathcal{P} = \{X_h\}_{h=1}^{\eta}$ of sets satisfying $X = \cup_{h=1}^{\eta} X_h$, with the interior $\text{int}(X_h) \neq \varnothing$ for all h and $\text{int}(X_h) \cap \text{int}(X_m) = \varnothing$ for $h \neq m$. The particular case $\eta = 1$, i.e., $\mathcal{P} = \{X\}$, is called *trivial partition* of X. If the sets X_h are simplices the partition is called a *simplicial partition* of X [19]. We denote by $\mathcal{V}(X_h)$ the set of vertices of the simplex X_h and by V_h the matrix having those vertices as columns. We can define a measure of the "fineness" of the simplicial partition $\mathcal{P}$ as

$$\delta(\mathcal{P}) \triangleq \max_{X_h\in\mathcal{P}} \max_{u,v\in\mathcal{V}(X_h)} \|u-v\|. \quad (1)$$

Consider the set $\mathcal{B}_1 = \{v \in \mathbb{R}^n : \|v\|_1 = 1\}$, $\|\cdot\|_1$ being the 1-norm of a vector, i.e., the sum of the absolute values of the vector components. The simplex $\mathcal{S} = \mathbb{R}^n_+ \cap \mathcal{B}_1$ is called *standard simplex*. It is always possible to find a simplicial partition of the standard simplex with $(n-1)$-simplices [19]. If the sets X_h are simplicial cones the partition is called a *simplicial conic partition* of X. We denote by $\{R_h\}_{h=1}^{\eta}$ the set of extremal ray matrices defining the cones of the simplicial conic partition of a cone $\mathcal{C}$. Note that given a simplicial conic partition of $\mathbb{R}^n_+$ the intersections of its cones with $\mathcal{B}_1$ uniquely provide a simplicial partition of the standard simplex $\mathcal{S} = \mathbb{R}^n_+\cap\mathcal{B}_1$, say $\mathcal{P}_\mathcal{S}$, and also the converse holds. More in general the following result holds.

***Lemma** 1:* Given a simplicial cone $\mathcal{C} \subset \mathbb{R}^n$ with extremal ray matrix R, and the standard simplex $\mathcal{S}$ in $\mathbb{R}^n$, any simplicial conic partition of $\mathcal{C}$ uniquely corresponds to a simplicial partition of the standard simplex and also the converse holds.

Proof: Consider a simplicial conic partition of $\mathcal{C}$ with corresponding extremal ray matrices $\{R_h\}_{h=1}^{\eta}$. Choose

$$V_h = R^{-1}R_h\Gamma_h, \quad h \in \{1,\ldots,\eta\} \quad (2)$$

where $R \in \mathbb{R}^{n\times n}$ is the matrix of extremal rays of a $\mathcal{V}$-representation of the cone $\mathcal{C}$ and Γ_h is the diagonal matrix with positive diagonal elements which ensures the columns of V_h having unitary 1-norm, i.e., the (j,j)-th element is given by $1/\|R^{-1}r_{h,j}\|_1$ for $j \in \{1,\ldots,n\}$ where $r_{h,j}$ is the j-th column of the matrix R_h. Then the columns of V_h are the vertices of the simplex X_h, defining for $h \in \{1,\ldots,\eta\}$ a simplicial partition of the standard simplex.

Now consider a simplicial partition of the standard simplex $\mathcal{S}$ with corresponding matrices of vertices $\{V_h\}_{h=1}^{\eta}$ and $V_h \in \mathbb{R}^{n\times n}$. Choose

$$R_h = RV_h, \quad h \in \{1,\ldots,\eta\}. \quad (3)$$

The invertibility of R ensures that (3) uniquely define a simplicial conic partition of $\mathcal{C}$. ■

Figure 1 illustrates a four cones partition of a cone $\mathcal{C} \subset \mathbb{R}^2$ and the corresponding simplicial partition of the standard simplex $\mathcal{S}$ through (3) in Lemma 1.

A symmetric matrix $M \in \mathbb{R}^{n\times n}$ is cone-copositive with respect to a cone $\mathcal{C} \subseteq \mathbb{R}^n$ if and only if $v^\top Mv \geq 0$ for any $v \in \mathcal{C}$. A cone-copositive matrix will be denoted by $M \succcurlyeq^{\mathcal{C}} 0$. If the equality only holds for $v = 0$, then M is *strictly cone-copositive* and the notation is $M \succ^{\mathcal{C}} 0$. In the particular case $\mathcal{C} = \mathbb{R}^n_+$, a (strictly) cone-copositive matrix is called (strictly) *copositive*, i.e., ($M \succ^{\mathbb{R}^n_+} 0$) $M \succcurlyeq^{\mathbb{R}^n_+} 0$. The notation $M \succcurlyeq 0$, i.e., without any superscript on the inequality, indicates that M is positive semidefinite. The cone-copositivity evaluation of a symmetric matrix M on a simplicial cone $\mathcal{C}$ can be always transformed into an equivalent copositive problem,

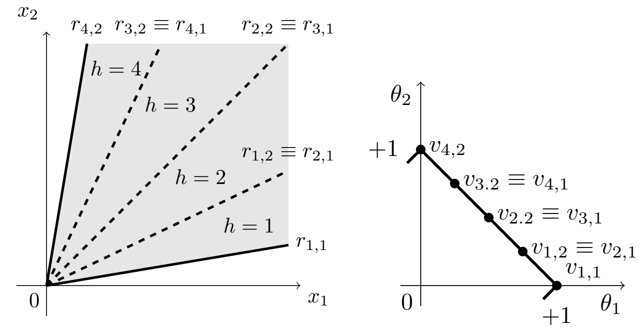

Fig. 1. A simplicial conic partition of a cone in $\mathbb{R}^2$ (gray area in the left figure) with the correspondent simplicial partition of the standard simplex (right), i.e., $R = [r_{1,1}\ r_{4,2}]$, $R_h = [r_{h,1}\ r_{h,2}]$, $V_h = [v_{h,1}\ v_{h,2}]$, with $h \in \{1,2,3,4\}$, see (3).

as stated by [20]. Moreover a sufficient condition for cone-copositivity in terms of LMIs can be derived.

***Lemma** 2:* Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $\mathcal{C}$ a simplicial cone with corresponding extremal ray matrix $R \in \mathbb{R}^{n\times n}$. Then

$$M \succ^{\mathcal{C}} 0 \quad (4)$$

if there exists a symmetric (entrywise) positive matrix N such that

$$R^\top MR - N \succcurlyeq 0. \quad (5)$$

Proof: From [20, Corollary 2.21] it follows that (4) is equivalent to

$$R^\top MR \succ^{\mathbb{R}^n_+} 0. \quad (6)$$

By applying Theorem 1 in [16] one has that (5) implies (6) and then the proof is complete. ■

The above result is valid also if the cone $\mathcal{C}$ is polyhedral but not simplicial.

A condition for not cone-copositivity of a matrix can be simply derived.

***Lemma** 3:* Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $\mathcal{C}$ a simplicial cone with extremal ray matrix $R \in \mathbb{R}^{n\times n}$. Then M is not cone-copositive with respect to $\mathcal{C}$ if

$$r^\top Mr < 0 \quad (7)$$

holds for some column r of R.

Proof: The proof directly follows from the definition of cone-copositivity. ■

III. Stability analysis

The stability result provided in this paper is based on the use of a continuous PWQ-LF defined over a conic partition of the state space.

Let us consider a partition of $\mathbb{R}^n$, say $\mathcal{P}_{\mathbb{R}^n}$, into a family of λ simplicial cones, i.e., a *simplicial complete fan* in $\mathbb{R}^n$. The cones are represented by

$$\mathcal{C}_i = \{R_i\theta,\ \theta \in \mathbb{R}^n_+\}, \quad i \in \{1,\ldots,\lambda\} \quad (8)$$

where $R_i \in \mathbb{R}^{n\times n}$ is a nonsingular matrix whose columns are the extremal rays of a $\mathcal{V}$-representation of $\mathcal{C}_i$.

A conewise linear system is described as follows

$$\dot{x} = A_i x \qquad \text{if } x \in \text{int}(\mathcal{C}_i), \tag{9a}$$

$$\dot{x} \in \overline{\text{co}}_{i\in\mathcal{I}(x)} \{A_i\}\, x \qquad \text{if } x \in \bigcap_{i\in\mathcal{I}(x)} \mathcal{C}_i, \tag{9b}$$

with $x \in \mathbb{R}^n$, $\mathcal{I}(x)$ the set of all indices $i \in \{1,\dots,\lambda\}$ such that $x \in \mathcal{C}_i$, $\{A_i\}_{i=1}^{\lambda}$ are known real matrices and $\{\mathcal{C}_i\}_{i=1}^{\lambda}$ are simplicial cones given by (8) which define the simplicial conic partition $\mathcal{P}_{\mathbb{R}^n}$.

Herein we adopt the following solution concept that allows sliding behavior.

***Definition** 1:* An absolutely continuous function $x(t) : \mathbb{R}_+ \to \mathbb{R}^n$ that satisfies the differential inclusion (9) almost everywhere is called Filippov solution [21].

It follows that system (9) has at least one solution for any initial condition x_0. We analyze the stability of the zero solution of (9) by using Lyapunov arguments.

A. Stability on the original partition

Consider a PWQ function in the form

$$V(x) = x^\top P_i x \quad \text{if } x \in \mathcal{C}_i, \quad i \in \{1,\dots,\lambda\}. \tag{10}$$

If there exist $\{P_i\}_{i=1}^{\lambda}$ matrices such that (10) is continuous, strictly positive and strictly decreasing in time along all solutions of (9), then the system is globally exponentially stable [4] and the function $V(x)$ is called a PWQ-LF for (9).

The continuity of (10) on the nontrivial cones intersections can be written as

$$x^\top P_i x = x^\top P_j x, \quad \forall x \in \mathcal{C}_i \cap \mathcal{C}_j, \tag{11}$$

for any $i, j \in \{1,\dots,\lambda\}$, such that $\mathcal{C}_i \cap \mathcal{C}_j \neq \{0\}$. Say $R_{ij} \in \mathbb{R}^{n\times m_{ij}}$, $m_{ij} < n$, the matrix of the common extremal rays of $\mathcal{C}_i$ and $\mathcal{C}_j$. Then if $x \in \mathcal{C}_i \cap \mathcal{C}_j$ one can write $x = R_{ij}\theta$ with $\theta \in \mathbb{R}_+^{m_{ij}}$. Therefore the continuity conditions (11) is guaranteed by

$$R_{ij}^\top (P_i - P_j) R_{ij} = 0 \tag{12}$$

for all $i, j \in \{1,\dots,\lambda\}$, such that $\mathcal{C}_i \cap \mathcal{C}_j \neq \{0\}$.

The search for a PWQ-LF can be formulated through the following result.

***Theorem** 1:* Consider the set of LMIs

$$R_i^\top P_i R_i - N_{P_i} \succcurlyeq 0 \tag{13a}$$

$$-R_i^\top (A_i^\top P_i + P_i A_i) R_i - N_{Q_i} \succcurlyeq 0 \tag{13b}$$

for all $i \in \{1,\dots,\lambda\}$ and

$$-R_{ij}^\top (A_j^\top P_i + P_i A_j) R_{ij} - N_{R_{ij}} \succcurlyeq 0 \tag{14}$$

for all $i, j \in \{1,\dots,\lambda\}$, such that $\mathcal{C}_i \cap \mathcal{C}_j \neq \{0\}$, together with the equality constraints (12). If (12)–(14) has a solution $\{P_i, N_{P_i}, N_{Q_i}, N_{R_{ij}}\}$ with $P_i \in \mathbb{R}^{n\times n}$ symmetric matrices, $N_{P_i} \in \mathbb{R}^{n\times n}$, $N_{Q_i} \in \mathbb{R}^{n\times n}$, $N_{R_{ij}} \in \mathbb{R}^{m_{ij}\times m_{ij}}$ symmetric positive matrices, then the conewise linear system (9) is globally exponentially stable, with (10) being a PWQ-LF.

Proof: Consider the function (10), with the P_i satisfying (12)–(14), as a candidate PWQ-LF. It is continuous thanks to conditions (12). Condition (13a) and Lemma 2 guarantee that such function is strictly positive in the cones. Condition (13b) and Lemma 2 ensure that it is also decreasing along the system trajectories when the state is in the cones interior. When the state is on the boundaries common to the cones, the function is strictly decreasing with respect to all the state trajectories that satisfy the differential inclusion (9b). Indeed, by using Lemma 2, the inequalities (13b) and (14) imply

$$-(A_i^\top P_i + P_i A_i) \succ^{\mathcal{C}_i \cap \mathcal{C}_j} 0, \tag{15a}$$

$$-(A_j^\top P_i + P_i A_j) \succ^{\mathcal{C}_i \cap \mathcal{C}_j} 0 \tag{15b}$$

for all $i, j \in \{1,\dots,\lambda\}$, such that $\mathcal{C}_i \cap \mathcal{C}_j \neq \{0\}$, respectively. Since (15) are valid for all $i, j \in \{1,\dots,\lambda\}$, by considering the convex combinations of conditions (15) one obtains that (10) is strictly decreasing also along trajectories lying on the boundaries common to cones. Thus the function (10) is a PWQ-LF for (9) which is globally exponentially stable. ■

B. Iterative refinement of the conic partition

Theorem 1 can be used also if the cones of the conewise linear system are polyhedral but not simplicial. In particular, each polyhedral cone can be partitioned into simplicial cones and, in order to define (9), the same dynamic matrix can be considered for all cones of its partition.

Analogously if it is not possible to determine a PWQ-LF for the system (9) with the given conic partition, one can restart with a new state space partition with a larger number of simplicial cones. The first natural trial for a PWQ-LF consists of considering the partition induced by the λ cones of the conewise linear system (9), so as shown in the previous subsection. We consider this as a first iteration for the PWQ-LF search. Say η such an iteration index. Then at the η-th iteration the state space partition is obtained by partitioning each cone $\mathcal{C}_i$ in (8) into η simplicial cones $\{\mathcal{C}_{i,h}^{\eta}\}_{h=1}^{\eta}$, i.e.,

$$\mathbb{R}^n = \bigcup_{i=1}^{\lambda} \mathcal{C}_i = \bigcup_{i=1}^{\lambda} \left(\bigcup_{h=1}^{\eta} \mathcal{C}_{i,h}^{\eta} \right). \tag{16}$$

Consider now a PWQ function in the form

$$V(x) = x^\top P_{i,h}^{\eta} x \quad \text{if } x \in \mathcal{C}_{i,h}^{\eta}, \tag{17}$$

with

$$\mathcal{C}_{i,h}^{\eta} = \{R_{i,h}^{\eta}\theta,\, \theta \in \mathbb{R}_+^n\}, \tag{18}$$

and $i \in \{1,\dots,\lambda\}$, $h \in \{1,\dots,\eta\}$. In other words at the first iteration ($\eta = 1$) the extremal ray matrices are given by

$$\{R_{i,h}^{\eta}\}_{h=1}^{\eta} \triangleq R_i, \quad \text{with } \eta = 1 \tag{19}$$

and $i \in \{1,\dots,\lambda\}$, which means that at the first iteration (17)–(18) coincide with (10) and (8), respectively.

For $n \geq 4$ it is not trivial to find a procedure for the selection of the cones of a new partition. In the following we propose an approach which exploits the so called *bisection* *along the longest edge* (BALE) technique of the standard simplex [22]. The proof of Lemma 1 shows that finding $\{R_{i,h}^{\eta}\}_{i=1,h=1}^{\lambda,\eta}$ corresponds to finding a suitable simplicial partition $\mathcal{P}_{\mathcal{S}}$ of the standard simplex $\mathcal{S}$, say $\mathcal{P}_{\mathcal{S}}^{\eta}$. Without loss of generality the same partition $\mathcal{P}_{\mathcal{S}}^{\eta}$ of the standard simplex can be considered for all cones $\mathcal{C}_i$: in this way the partitions of the different new cones will be congruent. In particular, by using Lemma 1 and the generalization of (3) one can write

$$R_{i,h}^{\eta} = R_i V_h^{\eta} \tag{20}$$

with $i \in \{1, \ldots, \lambda\}$, $h \in \{1, \ldots, \eta\}$, where V_h^1 is the identity matrix and V_h^{η} is the matrix of the vertices of the simplex X_h^{η} of $\mathcal{P}_{\mathcal{S}}^{\eta}$ at the η-th iteration.

Note that the BALE technique guarantees that the fineness $\delta(\mathcal{P}_{\mathcal{S}}^{\eta})$ goes to zero as the refinement steps proceed. It provides at each iteration an increase by one of the number of cones in the simplicial partition of $\mathcal{C}_i$ and then an increase of λ new cones in the simplicial partition of the state space. In order to carefully consider the sliding mode conditions, the cones of the new partitions of $\mathbb{R}^n$ generated at each iteration must be checked whether some of their faces belongs to a boundary of two "original" cones $\mathcal{C}_i$ and $\mathcal{C}_j$. In order to do that at the η-th iteration, for each i and for each j, one can define a matrix $R_{ij,h}^{\eta}$ given by the columns of $R_{i,h}^{\eta}$ which belong to the conic hull of some columns of the matrix R_{ij}. When $\mathcal{C}_{i,h}^{\eta} \cap \mathcal{C}_j = \{0\}$ such matrix is not defined.

Theorem 1 can be now extended to the case of refined conic partitions of the state space, i.e., to consider candidate PWQ-LFs with a larger number of "pieces" corresponding to many matrices $\{P_{i,h}^{\eta}\}_{h=1}^{\eta}$ for each system's cone $\mathcal{C}_i$. This is shown by the following result.

Theorem 2: Consider the set of LMIs

$$(R_{i,h}^{\eta})^{\top} P_{i,h}^{\eta} R_{i,h}^{\eta} - N_{P_{i,h}}^{\eta} \succcurlyeq 0 \tag{21a}$$

$$-(R_{i,h}^{\eta})^{\top} (A_i^{\top} P_{i,h}^{\eta} + P_{i,h}^{\eta} A_i) R_{i,h}^{\eta} - N_{Q_{i,h}}^{\eta} \succcurlyeq 0 \tag{21b}$$

for $i \in \{1, \ldots, \lambda\}$, $h \in \{1, \ldots, \eta\}$ and

$$-(R_{ij,h}^{\eta})^{\top} (A_j^{\top} P_{i,h}^{\eta} + P_{i,h}^{\eta} A_j) R_{ij,h}^{\eta} - N_{R_{ij,h}}^{\eta} \succcurlyeq 0 \tag{22}$$

for $i, j \in \{1, \ldots, \lambda\}$, and all $h \in \{1, \ldots, \eta\}$ such that the matrix $R_{ij,h}^{\eta}$ is defined so as described above, together with the equality constraints

$$(R_{ij,h\ell}^{\eta})^{\top} (P_{i,h}^{\eta} - P_{j,\ell}^{\eta}) R_{ij,h\ell}^{\eta} = 0 \tag{23}$$

for $i, j \in \{1, \ldots, \lambda\}$, $h, \ell \in \{1, \ldots, \eta\}$ where $R_{ij,h\ell}^{\eta}$ is the matrix of common extremal rays between $\mathcal{C}_{i,h}^{\eta}$ and $\mathcal{C}_{j,\ell}^{\eta}$ such that $\mathcal{C}_{i,h}^{\eta} \cap \mathcal{C}_{j,\ell}^{\eta} \neq \{0\}$. Then the conewise linear system (9) is globally exponentially stable if there exists an η, such that the set of LMIs (21)–(22) together with (23) has a solution $\{P_{i,h}^{\eta}, N_{P_{i,h}}^{\eta}, N_{Q_{i,h}}^{\eta}, N_{R_{ij,h}}^{\eta}\}$ with $P_{i,h}^{\eta} \in \mathbb{R}^{n \times n}$ symmetric matrices, $N_{P_{i,h}}^{\eta} \in \mathbb{R}^{n \times n}$, $N_{Q_{i,h}}^{\eta} \in \mathbb{R}^{n \times n}$, $N_{R_{ij,h}}^{\eta} \in \mathbb{R}^{m_{ij,h} \times m_{ij,h}}$ symmetric positive matrices.

Proof: The proof follows from the application of Theorem 1 to the state space conic partition corresponding to the η-th iteration. ■

Note that, analogously to conditions (14) which are considered only on the boundaries of the system's cones, the sliding conditions (22) are written only for the boundaries among the original cones.

The practical interest of the above result resides in the fact that any solution of (21)–(23) directly provides the matrices $\{P_{i,h}^{\eta}\}_{i=1,h=1}^{\lambda,\eta}$ of a PWQ-LF (17).

A necessary condition for the existence of a PWQ-LF can be obtained by considering the cone-copositivity definition without invoking the sliding conditions and the continuity.

Lemma 4: The conewise linear system (9) does not admit a PWQ-LF with the state space conic partition (16), if there exists a cone $\mathcal{C}_{i,h}^{\eta}$ with a corresponding extremal ray matrix $R_{i,h}^{\eta} \in \mathbb{R}^{n \times n}$, such that the inequalities

$$r^{\top} P_{i,h}^{\eta} r > 0 \tag{24a}$$

$$-r^{\top} (A_i^{\top} P_{i,h}^{\eta} + P_{i,h}^{\eta} A_i) r > 0 \tag{24b}$$

have no solution $P_{i,h}$ for some column r of $R_{i,h}^{\eta}$.

Proof: The proof easily follows from the definition of strict cone-copositivity. ■

IV. Examples

The procedure for finding a PWQ-LF for the conewise linear system (9) is implemented in Algorithm 1. Of course the increase of the total number of cones ($\lambda\,\eta$) is paid with the increase of the computational burden, especially for high order systems. The numerical results have been obtained by using `Matlab` and `CVX` [23], [24] with a PC Intel Dual Core processor at 2.8 GHz. All inequalities have been also a posteriori checked to be strictly verified with a lowest bound equal to 10^{-8} (note that the LMIs (21)–(22) are not strict).

Consider the Example 2 in [12] with $\delta = 0$ which can be written in the form (9) with

$$A_1 = \begin{pmatrix} -2 & -2 \\ 4 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -2 & 2 \\ -4 & 1 \end{pmatrix} \tag{25}$$

where A_1 holds for $x_2 \geq 0$ and A_2 for $x_2 \leq 0$. In Fig. 2 is reported the state trajectory and some level curves of the PWQ-LF obtained by Algorithm 1 with $\eta = 4$ cones given by the four quadrants. In Fig. 3 the time evolutions of the state variables which present a sliding behavior are shown. Note that the PWQ-LF, which decreases also on the sliding modes, has discontinuities in its time derivative when the state variables change their sign, which is coherent with the conic state space partition into the four quadrants, see also the level curves in Fig. 2.

Consider the Example 1 in [12] which can be written in the form (9) with

$$A_1 = \begin{pmatrix} 1 & -2 \\ 2 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix} \tag{26}$$

where A_1 holds for $x_2 \geq 0$ and A_2 for $x_2 \leq 0$. Without the conditions (22) for inclusion of sliding modes dynamics, the problem (21) with (23) by considering a partition with four cones given by the quadrants would be feasible, providing a wrong conclusion about the stability of the system, analogously to what is shown in [7], [12]. By including the conditions (22) the problem becomes infeasible.

Algorithm 1: Algorithm for finding a PWQ-LF

Data: $\{A_i, R_i\}_{i=1}^{\lambda}$, $\eta_{\max}$, i.e., the conewise linear system and the maximum number of elements for $\mathcal{P}_{\mathcal{S}}$

Result: "No PWQ-LF function exists" or "A PWQ-LF (17) exists with $\{P_{i,h}^{\eta}\}_{i=1,h=1}^{\lambda,\eta}$ found" or "$\eta_{\max}$ has been reached"

begin
- $n \leftarrow$ `size`(A_i);
- $\eta \leftarrow 1$;
- $V_1 \leftarrow I_n$;
- **repeat**
  - `/* Compute the extremal ray matrices of the partition */`
  - **for** $i \leftarrow 1$ **to** λ **do**
    - **for** $j \leftarrow 1$ **to** $\lambda,\ j \neq i$ **do**
      - **for** $h \leftarrow 1$ **to** η **do**
        - $R_{i,h}^{\eta} \leftarrow R_i V_h$;
        - $R_{ij,h}^{\eta} \leftarrow$ all columns of $R_{i,h}^{\eta}$ in the conic hull of R_{ij};
        - **for** $\ell \leftarrow 1$ **to** η **do**
          - $R_{j,\ell}^{\eta} \leftarrow R_j V_\ell$;
          - $R_{ij,h\ell}^{\eta} \leftarrow$ all common columns of $R_{i,h}^{\eta}$ and $R_{j,\ell}^{\eta}$;
        - **end**
      - **end**
    - **end**
  - **end**
  - necCond $\leftarrow$ `feasible` (24);
  - **if** necCond **then**
    - `/* Feasibility for a PWQ-LF */`
    - suffCond $\leftarrow$ `feasible` (21)–(23);
    - **if** suffCond **then**
      - `/* Compute the solution */`
      - $\{P_{i,h}^{\eta}, N_{P_{i,h}}^{\eta}, N_{Q_{i,h}}^{\eta}, N_{R_{ij,h}}^{\eta}\}_{i=1,h=1}^{\lambda,\eta} \leftarrow$ `getSolution` (21)–(23);
    - **else**
      - `/* BALE refinement of the standard simplex */`
      - $\{\bar{h}, v_{\bar{h},l}, v_{\bar{h},m}\} \leftarrow$ `maxDistanceColumn` $(\{V_h\}_{h=1}^{\eta})$;
      - $w \leftarrow \frac{1}{2}(v_{\bar{h},l} + v_{\bar{h},m})$;
      - $V'_{\bar{h}} \leftarrow [v_{\bar{h},1} \cdots v_{\bar{h},l-1}\, w\, v_{\bar{h},l+1} \cdots v_{\bar{h},n}]$;
      - $V''_{\bar{h}} \leftarrow [v_{\bar{h},1} \cdots v_{\bar{h},m-1}\, w\, v_{\bar{h},m+1} \cdots v_{\bar{h},n}]$;
      - $\{V_h\}_{h=1}^{\eta+1} \leftarrow \{V_1, \ldots, V_{\bar{h}-1}, V'_{\bar{h}}, V''_{\bar{h}}, V_{\bar{h}+1}, \ldots, V_\eta\}$;
      - $\eta \leftarrow \eta + 1$;
    - **end**
  - **end**
- **until** *(NOT(*necCond*) OR* suffCond *OR* $\eta > \eta_{\max}$*)*;

end

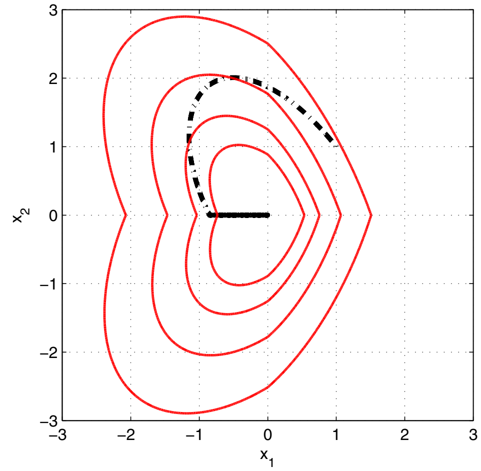

Fig. 2. State space: system trajectory (dash-dotted line) and level curves of the resulting PWQ-LF for the system (9) with (25), which exhibits sliding behavior.

The same result is also obtained by considering

$$A_1 = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 5 \\ -1 & -3 \end{pmatrix} \tag{27}$$

which are taken from Example 1.1 in [25] which exhibits unstable sliding behavior.

Algorithm 1 can be also adopted for systems which do not exhibit sliding. In particular consider the switched system with dynamic matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \gamma & -1 \\ 1 & \gamma \end{pmatrix} \tag{28}$$

such that

$$\dot{x} = A_1 x \quad \text{if } x_1(x_2 - h(x_1)) \leqslant 0 \tag{29a}$$
$$\dot{x} = A_2 x \quad \text{if } x_1(x_2 - h(x_1)) \geqslant 0, \tag{29b}$$

where $h(x_1) = -x_1/(\gamma + \sqrt{\gamma^2 + 1})$. The system (29) does not admit any convex Lyapunov function [26]. By choosing $\gamma = 1.2$, Algorithm 1 provides a PWQ-LF with 20 cones. In Fig.s 4–5 the corresponding state space trajectory and time evolutions are shown. Note that the trajectory remains for long time very close to the $x_1 = 0$ axis without sliding.

We checked Algorithm 1 also for the fourth order switched system given by Example 7 in [27], which does not admit a PWQ-LF with the original two halfspaces state space partition, so as shown in that paper; our procedure provides a PWQ-LF with a state space partition into 16 cones.

V. Conclusions

Conditions for the existence of a PWQ-LF and hence for the exponential stability of conewise linear systems with sliding modes have been provided. An iterative state space partition algorithm is proposed for the search of a PWQ-LF. Examples show the validity of the proposed approach in the presence or not of stable and unstable sliding modes. Future work will be dedicated to the extension of the results to the case of piecewise affine systems.

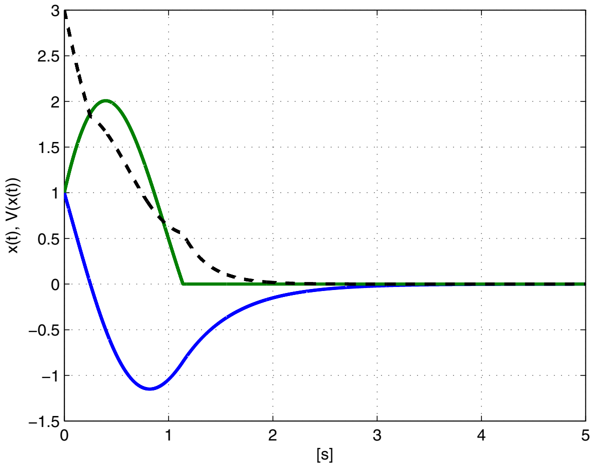

Fig. 3. Time evolutions of the two state variables and PWQ-LF (dashed line) computed along the system trajectory for the system (9) with (25).

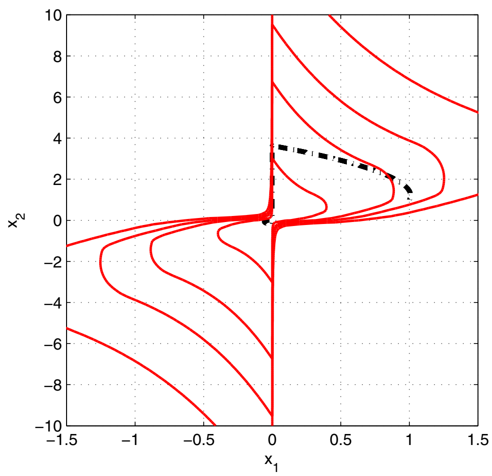

Fig. 4. State space: system trajectory (dash-dotted line) and level curves of the resulting PWQ-LF for the system (28)–(29).

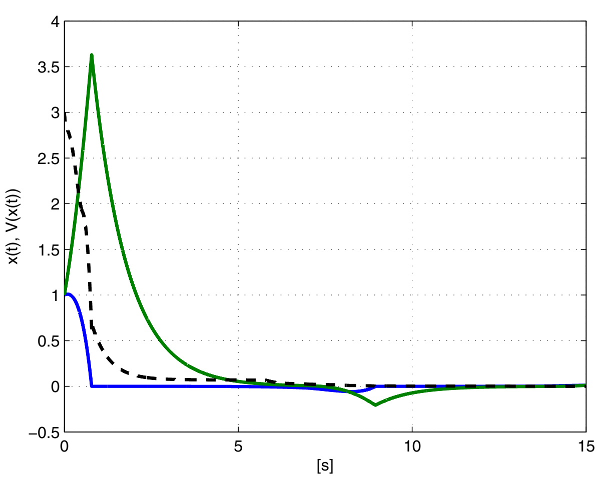

Fig. 5. Time evolutions of the two state variables and PWQ-LF (dashed line) computed along the system trajectory for the system (28)–(29).

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