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Averaging for switched DAEs

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Switched differential-algebraic equations (switched DAEs) $E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t)$ are suitable for modeling many practical systems, e.g. electrical circuits. When the switching is periodic and of high frequency, the question arises whether the solutions of switched DAEs can be approximated by an average non-switching system. It is well known that for a quite general class of switched ordinary differential equations (ODEs) this is the case. For switched DAEs, due the presence of the so-called consistency projectors, it is possible that the limit of trajectories for faster and faster switching does not exist. Under certain assumptions on the consistency projectors a result concerning the averaging for switched DAEs is presented.

1 Introduction

The basic idea of averaging for switched systems is that for sufficiently fast switching the solutions of the switched system approach the solutions of a non-switched system—the average system. In particular, if the switching is much faster than the dynamics of each mode then this approach is suitable to study system properties (e.g. stability, reference tracking) of the switched system via the simpler average system. Applications of this approach can be found in pulse-width modulation, sliding mode controllers or in general when fast digital controllers are applied to a relatively slow physical system. For a recent overview of the averaging technique see [1,2].

For a switched linear ODE

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad x(0) = x_0 \tag{1}$$

with periodic switching signal $\sigma : \mathbb{R} \to \{1, 2, \ldots, M\}$ it is easy to see that the following general averaging result holds.

Theorem 1.1 ([3]) *Let the average system be given by*

$$\dot{x}_{\text{av}} = A_{\text{av}}x_{\text{av}}, \quad x_{\text{av}}(0) = x_0 \tag{2}$$

with $A_{\text{av}} := d_1A_1 + d_2A_2 + \ldots + d_MA_M$ *where* d_k *denotes the duty cycle of the* k*-th mode,* $k \in \{1, \ldots, M\}$*, i.e.* d_k *is the fraction of the period* $p > 0$ *of the switching signal for which mode* k *is active. Then on every compact time interval* $[0, T]$*:*

$$\|x(t) - x_{\text{av}}(t)\| = O(p).$$

where x *and* x_{av} *denotes the solutions of* (1) *and* (2)*.*

Modeling of electrical circuits yields switched differential-algebraic equations (DAEs) [4]

$$E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t) \tag{3}$$

and the question occurs whether the above averaging result can be generalized to this system class. The following example shows that this is *not* the case in general.

Example 1.2 (A counterexample) Consider the switched DAE (3) with $M = 2$ and

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}\right), \; (E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right).$$

The solutions are shown in Figure 1 and it is apparent that the jumps induced by the switching prevents convergence to a single trajectory for faster switching.

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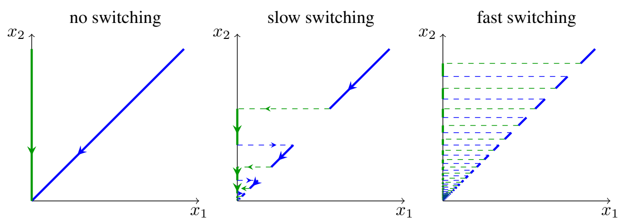

Fig. 1: Counter example. Jumps are indicated by dashed lines.

For the forthcoming analysis of (3) we will assume that the switching signal σ is periodic with period $p > 0$. Our goal is to study the solution behavior of the switched DAE with decreasing switching period p but with constant duty cycles $d_1, d_2, \ldots, d_M > 0$. Furthermore we assume that the matrix pairs (E_k, A_k), $k \in \{1, 2, \ldots, M\}$, are regular, i.e. $\det(sE_k - A_k) \not\equiv 0$ and we assume that the switched system is impulse free [5,6] (that does not exclude jumps!). Under these assumptions there exists unique consistency projectors Π_k and flow matrices A_k^{diff}, $k \in \{1, \ldots, M\}$ such that the solutions of the switched DAE have the form [7,8]

$$x(t) = e^{A_i^{\text{diff}}(t-t_i)}\Pi_i e^{A_{i-1}^{\text{diff}}(t_i-t_{i-1})}\Pi_{i-1}\cdots$$
$$\cdots e^{A_2^{\text{diff}}(t_3-t_2)}\Pi_2 e^{A_1^{\text{diff}}(t_2-t_1)}\Pi_1 x(t_1-).$$

where, t_j, $j = 1, 2, \ldots, i$, are the switching times in the interval $[0, t)$; for ease of notation we assumed a strictly increasing mode sequence and identified j with $j - M$ for $j > M$.

2 Avaraging result for switched DAEs

In our recent paper [9] we have established an averaging result for the case $M = 2$. The crucial additional assumption we have to make is commutativity of the consistency projectors. We were able to generalize this result to more than two modes [10], but the proof methods are rather different and more involved in the general case. However, the (pairwise) commutativity of the consistency projectors remains the crucial assumption:

$$\forall i, j \in \{1, \ldots, M\} : \quad \Pi_i\Pi_j = \Pi_j\Pi_i. \tag{4}$$

Note that commutativity of the consistency projectors follows from the assumption that the flow matrices A_k^{diff} commute [11]. Under this stronger assumption it is quite simple to prove the forthcoming averaging result for switched DAEs, but we only assume the weaker condition (4) here. A crucial consequence of (4) is the following result

Lemma 2.1 ([10]) *If* (4) *holds then*

$$\operatorname{im}\Pi_1\Pi_2\cdots\Pi_M = \operatorname{im}\Pi_1 \cap \operatorname{im}\Pi_2 \cap \ldots \cap \operatorname{im}\Pi_M$$

Because of the previous lemma it makes sense to introduce the projector

$$\Pi_\cap := \Pi_1\Pi_2\cdots\Pi_M$$

which projects onto the intersection of the individual consistency spaces of each DAE mode. In fact, it is quite clear that the average system most evolve within this intersection if it exists. Note that in the above counter example we had

$$\Pi_1\Pi_2 = \Pi_1 \neq \Pi_2 = \Pi_2\Pi_1.$$

We are now ready to state our main result concerning averaging for switched DAEs whose proof can be found in [10].

Theorem 2.2 (Averaging for switched DAEs) *Consider the impulse free regular switched DAE* (3) *with flow matrices* $A_1^{\text{diff}}, \ldots, A_M^{\text{diff}}$ *and consistency projectors* $\Pi_1, \ldots, \Pi_M$ *satisfying* (4)*. Then the average system*

$$\dot{x}_{\text{av}} = \Pi_\cap A_{\text{av}}^{\text{diff}} \Pi_\cap x_{\text{av}}, \quad x_{\text{av}}(0) = \Pi_\cap x(0-)$$

where $A_{\text{av}}^{\text{diff}} := d_1 A_1^{\text{diff}} + d_2 A_2^{\text{diff}} + \ldots + d_M A_M^{\text{diff}}$ *satisfies*

$$\|x(t) - x_{\text{av}}(t)\| = O(p) \quad \forall t \in (0, T].$$

We illustrate the theoretical result with the following example based on a simple electrical circuit.

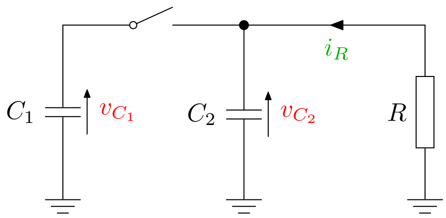

Fig. 2: A simple switching electrical circuit

Example 2.3 Consider the electrical circuit as shown in Figure 2. Let the state variable be $x = (v_{C_1}, v_{C_2}, i_R)^\top$, then it is easily seen, that the dynamics for the open switch can be described by $E_1\dot{x} = A_1 x$ where

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 & 0 \\ C_1 & 0 & 0 \\ 0 & C_2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & -R \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right),$$

and for the closed switch the dynamics are given by

$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 & 0 \\ C_1 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -R \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \right).$$

The corresponding consistency projectors are

$$\Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{R} & 0 \end{bmatrix}, \quad \Pi_2 = \frac{1}{C_1 + C_2} \begin{bmatrix} C_1 & C_2 & 0 \\ C_1 & C_2 & 0 \\ \frac{C_1}{R} & \frac{C_2}{R} & 0 \end{bmatrix}$$

and (4) holds:

$$\Pi_1\Pi_2 = \Pi_2 = \Pi_2\Pi_1.$$

Hence for sufficiently fast switching any solution can be approximated by a solution of the average system, although the solution of the switched DAE still exhibits jumps. This behavior is illustrated in Figure 3 for the duty cycle $d_1 = 0.4$ (and therefore $d_2 = 0.6$) and period $p = 0.02$.

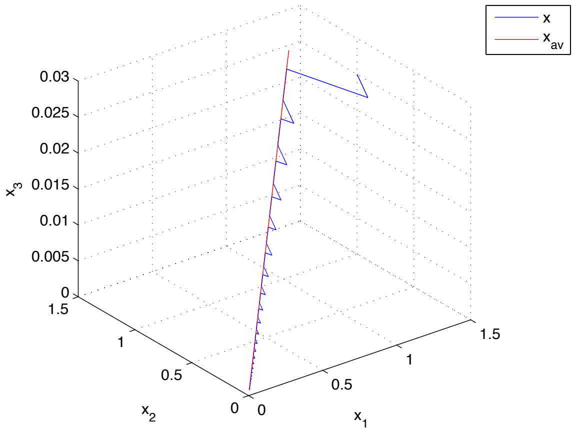

Fig. 3: Simulation result for the electrical circuit example.

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