Computation of steady state oscillations in power converters through complementarity

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***Abstract*—Computation of periodic steady state in nonlinear circuits is a key issue. Power electronics converters represent an interesting class of switched nonlinear circuits. The behavior of the converter is obtained by the commutations of the electronic devices which determine the switchings among the different converter modes. Switchings can be classified as external, if forced by directly manipulable control variables, and internal if determined by state dependent conditions. The presence of internal switchings makes difficult to know a priori the sequence of modes and also open loop steady state behaviors are difficult to be obtained. In this paper the complementarity modeling framework is proposed as a possible approach for computing periodic steady state oscillations in power converters with internal switchings. It is shown how linear complementarity systems can be used to model the behavior of a wide class of power converters. The discretization of such model allows to formulate a static complementarity problem whose solution provides the steady state oscillation of the converter. It is proved that backward Zero–Order–Hold technique preserves passivity through the discretization and allows to determine the unique solution of the complementary problem. A resonant converter and a dc/dc voltage–mode controlled buck converter are used as examples.**

I. Introduction

Most power converters can be viewed as electrical networks with linear elements (resistors, inductors, capacitors, transformers), voltage and current sources, and electronic devices such as diodes and electronic switches (thyristors, transistors). A classical approach for modeling power converters consists of idealizing diodes and switches with short circuits, open circuits or similar characteristics, discriminating among the different modes of the converter, building for each mode a linear–time–invariant dynamic model and determining the conditions for the commutations among the different modes [1], [2]. The resulting model is usually called a *switched* model. The commutations of the electronic devices are typically classified as externally controlled, when an independent signal determines the state of the switch, or internally controlled. Internally controlled commutations can be due to closed loop controlled switches or to change in diodes state from conducting to blocking or vice-versa. The discontinuous conduction mode is a typical example of the latter class. In the presence of internally controlled commutations the switched model eventually becomes rather complex also for simple converter topologies [1], [3]. In this paper we deal with the computation of steady state oscillation of known period (not necessarily equal to the switching period) for power converters with internally controlled commutations. The class of converters considered can contain diodes and controlled switches, such that the converter can be represented as a circuit with diodes and state dependent (possibly discontinuous) inputs.

The model complexity in the presence of internally controlled commutations predicts the difficulties for knowing a priori the sequence of modes, also in the steady state. Using time–stepping simulations and ‘waiting’ for possible steady state is often not practical because in most cases the time constants of the modes are much larger than the switching period. The problem of finding directly the steady state solution for smooth nonlinear or piecewise-linear circuits has been investigated since tents of years ago and it is still of interest [4]. Dealing with power converters the solutions proposed in the literature can be classified as frequency–domain methods and time–domain methods. Examples of frequency–domain approaches are fundamental mode approximation [5], harmonic balance [6], [7], describing function [8]. By using their basic formulations, these approaches are very useful in order to obtain an estimation of the oscillation, but may become too computationally expensive if high order harmonics are involved in the solution. Within the time–domain framework most techniques are based on the classical shooting method which is modified in order to make it suitable for switched systems. Several papers have been published on that type of solution, already before 1990, see [9] and the references therein, and also in the next decade, see [10] and the references therein. One of the main problems when applying the shooting method to power converters consists of the knowledge of the sequence of modes in steady state, or in other words on the evaluation of the Jacobian for the steady state map. Usually the Jacobian is numerically obtained through time–stepping simulations over one period, starting from a guess of the state variables at steady state [10]–[12]. The Jacobian computation is a key issue for time–domain approaches when applied to switched models. This is also demonstrated by several papers dedicated to the sensitivity analysis of the switching times with respect to variations of the initial state guess, see among others [9], [13], [14]. The Jacobian computation problem becomes also more critical in some specific situations: when a small variation of the initial state causes a change in the sequence of modes, e.g. around the boundary between continuous and discontinuous conduction modes; for the computation of unstable steady state orbits; in the presence of state jumps. Within the literature on steady state for power converters one should also mention some interesting contributions dealing with specific issues: solutions for autonomous converters where the oscillation

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period is unknown [11], [15], [16]; numerical efficiency by parallel processing [17]; minimization of the number of state variables via topological analysis [18]. Some other solutions are dedicated to specific topologies [19], [20] or are based on equivalent impedances models of the switches [21], [22].

In this paper, by using both frequency–domain and time–domain ingredients, we propose the use of the complementarity formalism [23] for computing periodic oscillations for a wide class of power converters. Complementarity systems have been used to model (dynamic) switched electrical networks that contain ideal diodes and ideal switches [24], [25]. More recently the complementarity formalism has been proposed as an interesting alternative for obtaining complete switched models of power electronics converters [26]. The complementarity model is simple to be built and captures all modes of the converter, without enumerating them, nor assuming a priori knowledge of the sequence of modes and of the switching time commutation instants. Classical techniques usually allow to know the sequence of modes, but that might depend, also for simple topologies, on the parameters and inputs values. Instead, such information is not used at all for the construction of the proposed complementarity model. For example if some parameter of the converter is modified making the converter operating in continuous or discontinuous conduction mode, the sequence order of modes changes (we have one more mode when some current goes to zero during a period of the steady state oscillation) but the proposed technique is still able to find the steady state periodic oscillation without modifying anything in the model as well as in the algorithm. The construction of the complementarity model requires only the application of Kirchhoff Voltage and Current Laws on the assigned topology, similarly to what happens for the model construction of a generic electrical circuit by using the classical modified nodal analysis approach. On the other hand, the model of each mode can be obtained through simple algebraic transformations on the general complementarity model [26], [27]. The converter complementarity model is used to reformulate the problem of the computation of a periodic steady state oscillation as a static linear complementarity problem. We also prove that if the circuit is passive and consistency of the model discretization holds, the solution of the complementarity problem and the corresponding periodic oscillation are unique. The proposed approach is tested for the computation of the periodic steady state oscillations in a LLC resonant converter [28]–[31] and for computing multiple periodic oscillations (stable and unstable) of a voltage–mode controlled dc/dc buck converter.

II. Complementarity models of power converters

Before presenting some examples of complementarity models of power converters, we introduce some useful definitions.

***Definition** 1:* Given a real vector q and a real matrix M, a *linear complementarity problem* (LCP) consists of finding a real vector z such that

$$z \geqslant 0 \tag{1a}$$
$$q + Mz \geqslant 0 \tag{1b}$$
$$z^T(q{+}Mz) = 0, \tag{1c}$$

where the inequalities are considered componentwise.

In the sequel conditions (1) that define the LCP(q,M) will be more compactly indicated by means of the complementarity condition

$$w = q + Mz \tag{2a}$$
$$0 \leqslant w \perp z \geqslant 0. \tag{2b}$$

Then $\perp$ is the orthogonality symbol, i.e. given two real vectors z and w the notation $w \perp z$ stands for $z^T w = 0$ (the scalar product is zero). The relation (2b) implies that for each pair of scalar complementarity variables at least one of them must be zero.

It can be shown that the LCP(q,M) has a unique solution for any q if and only if M is a P-matrix [23]. A matrix M is called a P-matrix if all its principal minors are strictly positive. According to the definition, every positive definite matrix is a P-matrix but the converse is not true. Therefore being M a positive definite matrix implies uniqueness of the LCP(q,M) solution.

We now introduce the concept of a complementarity system.

***Definition** 2:* A *continuous–time linear complementarity system* (LCS) is the following linear system subject to complementarity constraints on z and w variables:

$$\dot{x} = A_c x + B_c z + E_c u \tag{3a}$$
$$w = C_c x + D_c z + F_c u \tag{3b}$$
$$0 \leqslant w \perp z \geqslant 0, \tag{3c}$$

where $x \in \mathbb{R}^{N_x}$ is the state vector, $u \in \mathbb{R}^{N_u}$ is an exogenous input vector, $z \in \mathbb{R}^{N_z}$ and $w \in \mathbb{R}^{N_z}$ are the complementarity variables, and A_c, B_c, C_c, D_c, E_c, F_c are real matrices of suitable dimensions.

In order to present the converters complementarity models, it is important to describe the behavior of the so–called ideal diode (ID) within the complementarity framework. It is straightforward that the current–voltage characteristic of an ID shown in Fig. 1 can be represented by means of the complementarity condition (3c). Note that the complementarity condition (3c) is still representative of the ID characteristic also by choosing w as the ID current and z as the ID voltage. The topology of the circuit dictates whether a diode current should be denoted as z (and thus affecting the time derivative of a state variable) or w (and thus not entering in the dynamic equations of the state but satisfying just algebraic equations).

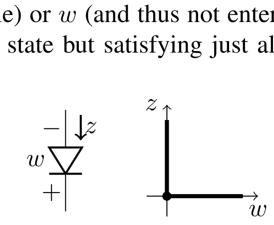

Fig. 1. Ideal diode symbol with the corresponding current–voltage characteristic and the indication of a possible pair of complementarity variables. Note that by choosing z as the voltage and w as the current, the characteristic maintain the same (complementarity) representation.

In what follows we consider two power converters, whose models can be represented as LCSs and for which it is interesting to analyze their cyclic steady state behaviors.

A. Resonant converter

A typical LLC resonant dc/dc converter is shown in Fig. 2 [28], [29], [32]. The two ideal switches S_1 and S_2 are controlled in anti–phase with a switching frequency f_s. The modulation determines a square wave voltage v_{in} with amplitude V_{dc}. The capacitor C_1 together with the inductors L_1 and L_2 represent a resonant circuit. A transformer with center tapped secondary is used to connect the resonant circuit to a diode rectifier. If $|L_2\dot{x}_3| < nx_4$ the converter will start operating in discontinuous conduction mode. For a more detailed description of the circuit operations see [28], [29].

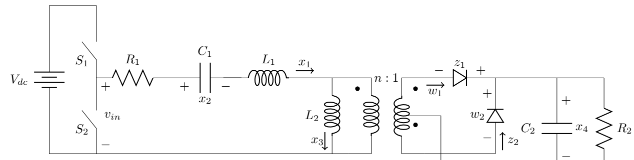

Fig. 2. Circuit scheme of a dc/dc LLC resonant converter.

By applying the Kirchhoff laws to the circuit in Fig. 2 and by using the ideal transformer relations it follows:

$$L_1\dot{x}_1 = -R_1x_1 - x_2 - L_2\dot{x}_3 + v_{in} \tag{4a}$$
$$C_1\dot{x}_2 = x_1 \tag{4b}$$
$$C_2\dot{x}_4 = -\frac{1}{R_2}x_4 + w_1 + z_2 \tag{4c}$$
$$w_2 = z_1 + \frac{2}{n}L_2\dot{x}_3 \tag{4d}$$

which must be satisfied independently of the commutations of the switches and independently of the conducting or blocking states of the diodes. From the voltage and current characteristics of the ideal transformer it follows

$$L_2\dot{x}_3 = n(x_4 - z_1) \tag{5a}$$
$$x_1 - x_3 = \frac{1}{n}(w_1 - z_2). \tag{5b}$$

By combining (4) and (5) the model of the resonant converter becomes

$$L_1\dot{x}_1 = -R_1x_1 - x_2 - nx_4 + nz_1 + v_{in} \tag{6a}$$
$$C_1\dot{x}_2 = x_1 \tag{6b}$$
$$L_2\dot{x}_3 = nx_4 - nz_1 \tag{6c}$$
$$C_2\dot{x}_4 = nx_1 - nx_3 - \frac{1}{R_2}x_4 + 2z_2 \tag{6d}$$
$$w_1 = nx_1 - nx_3 + z_2 \tag{6e}$$
$$w_2 = 2x_4 - z_1. \tag{6f}$$

By using the IDs characteristics $0 \leqslant w_i \perp z_i \geqslant 0$ with $i = 1, 2$, and by choosing $u = v_{in}$ the model (6) can be simply recast in the form (3) with matrices given in the Appendix A.

In ordinary operating conditions the input v_{in} is periodic of period $T_s = 1/f_s$. In order to obtain the control–to–output frequency response the input v_{in} can be chosen to be periodic of period αT_s, with a suitable integer α [5], [8].

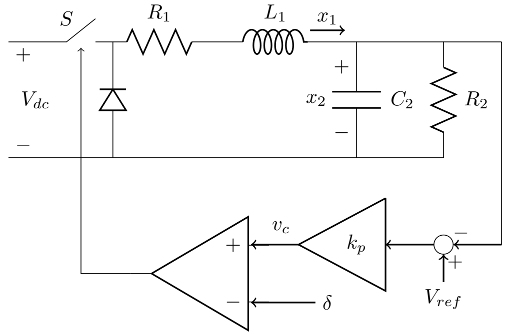

Fig. 3. Buck converter with PWM voltage–mode control.

B. Voltage–mode controlled buck converter

The complementarity model (3) is able to capture the behavior of some power converters topologies also under closed loop operating conditions. A dc/dc buck converter with voltage-mode pulse width modulation control is depicted in Fig. 3. Under normal operating conditions the behavior of the controlled buck converter can be represented by means of the circuit reported in Fig. 4. The input voltage v_{in} is imposed by the feedback control. In particular

$$v_{in} = V_{dc}\,\text{step}\,(v_c - \delta) \tag{7a}$$
$$\delta(t) = \frac{\Delta}{T_s} \mod (t, T_s), \tag{7b}$$

where δ is a sawtooth signal with amplitude Δ and period T_s and v_c is the control voltage. The set–valued function (7a) is defined as follows: $v_{in} = V_{dc}$ for $v_c > \delta$, $v_{in} = 0$ for $v_c < \delta$, and $v_{in} \in [0, V_{dc}]$ for $v_c = \delta$. That function can be represented in the following complementarity form

$$v_{in} = -V_{dc}z_2 + V_{dc} \tag{8a}$$
$$w_2 = z_3 + v_c - \delta \tag{8b}$$
$$w_3 = -V_{dc}z_2 + V_{dc} \tag{8c}$$
$$0 \leqslant w_2 \perp z_2 \geqslant 0 \tag{8d}$$
$$0 \leqslant w_3 \perp z_3 \geqslant 0. \tag{8e}$$

The complementarity variables z_2, z_3 and w_2, w_3 have not been marked in Fig. 3 and Fig. 4 since they are not physical variables, but they have been introduced in order to model the step function within the complementarity framework. Equations (8) can be simply explained. Indeed if $v_c > \delta$, from (8b) it follows that w_2 must be strictly positive independently of z_3 (zero or positive). Since the product between w_2 and z_2 must be zero, it follows that $z_2 = 0$ and from (8a) we get $v_{in} = V_{dc}$. If $v_c < \delta$, from (8b) it follows $z_3 > 0$ and then from (8e) it must be $w_3 = 0$, then from (8c) $z_2 = 1$ and $v_{in} = 0$. Finally if $v_c = \delta$, from (8c) and (8d) it follows that z_2 can take any value between 0 and 1 and then v_{in} can take any value between 0 and V_{dc}.

From Fig. 3 we can write

$$v_c = k_p\left(V_{ref} - x_2\right). \tag{9}$$

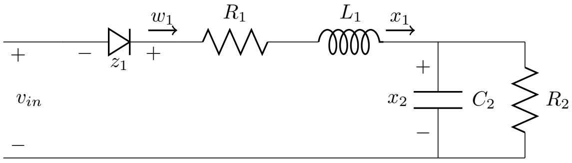

Fig. 4. Equivalent circuit of a controlled dc/dc buck converter.

By applying the Kirchhoff laws to the circuit in Fig. 3:

$$L_1\dot{x}_1 = -R_1x_1 - x_2 + z_1 + v_{in} \tag{10a}$$

$$C_2\dot{x}_2 = x_1 - \frac{1}{R_2}x_2 \tag{10b}$$

$$w_1 = x_1. \tag{10c}$$

By using (10)–(9) it is simple to obtain the model (3) with $x = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T$, $u = \begin{pmatrix} 1 & \delta \end{pmatrix}^T$, and the suitable matrices given in Appendix A.

Due to the external periodic signal δ (it is the PWM carrier signal or, equivalently, the dither signal [33]), a forced oscillation is expected. With the aim of obtaining the control–to–output frequency response the input v_c can be chosen to be periodic of period αT_s, with a suitable integer α [32].

III. Periodic steady state oscillations

In order to analyze the periodic steady state oscillations exhibited by systems in the form (3), one should first state existence and uniqueness of such type of solutions. A well-posedness analysis for (3) is out of the scope of this paper. On the other hand some considerations on that are in order. To analyze well-posedness one should first fix the solution concept. In [25] it has been shown that under the passivity assumption on (A_c, B_c, C_c, D_c) and for Bohl-type input signals u, i.e. signals that have a rational and strictly proper Laplace transform, under some technical assumptions it is possible to prove the existence of a so-called *forward solution* for (3). For a large class of power converters it is simple to show that the passivity hypothesis holds, but the typical inputs u to be considered are piecewise continuous which in general, clearly, are not Bohl-type signals. Therefore in order to use the forward solution concept one should extend the results in [25] to the case of piecewise continuous input signals. A contribution in this direction can be found in [34]. Instead, for our analysis we use the concept of *weak solution* [27].

Definition 3: Given an input $u(t)$, by a *weak solution* of (3) we mean a pair of trajectories $(x(t), z(t))$ such that $x(t)$ is absolutely continuous, $z(t)$ is integrable on $[0, T]$ and

$$x(t) - x(0) = \int_0^t \left[A_cx(\tau) + B_cz(\tau) + E_cu(\tau)\right] d\tau \tag{11a}$$

$$0 \leqslant C_cx(t) + D_cz(t) + F_cu(t) \perp z(t) \geqslant 0, \tag{11b}$$

for all $t \in [0, T]$.

Note that by using the weak solution concept we also exclude possible inconsistent initial conditions [35] and state jumps. In what follows we use the weak solution concept in order to compute possible cyclic steady state oscillations exhibited by the class of switched electronic systems under investigation.

A. Cyclic steady state behavior

Assume that given a periodic input $u(t)$ of period T, the system (3) has a periodic weak solution, i.e. a weak solution $(x(t), z(t))$ satisfying (3) for almost every time instant t and such that $x(t) = x(t+T)$, $z(t) = z(t+T)$ for every t, with T being the period of the solution. In the sequel a periodic weak solution will be also indicated as a periodic (steady state) oscillation. Under the above assumptions we can write

$$x(t) = e^{A_ct}x(0) + \int_0^t e^{A_c(t-\tau)}\left(B_cz(\tau) + E_cu(\tau)\right) d\tau, \tag{12}$$

and, being $x(t+T) = x(t)$, we can write

$$x(T) = e^{A_cT}x(0) + \int_0^T e^{A_c(T-\tau)}\nu(\tau)d\tau = x(0), \tag{13}$$

where for the sake of notation we used the position

$$\nu := B_cz + E_cu. \tag{14}$$

Then from (13)

$$x(0) = \left(I - e^{A_cT}\right)^{-1} \int_0^T e^{A_c(T-\tau)}\nu(\tau)d\tau. \tag{15}$$

Clearly, if A_c has some eigenvalue equal to zero, the periodic solution will not be unique. On the other hand, if A_c has all eigenvalue different from zero we are not sure on the uniqueness of the periodic solution because the initial condition of the steady state solution depends on the external input u and on the complementarity variable z which is unknown. By substituting (15) in (12) it follows that for any $t \in [0, T]$

$$x(t) = e^{A_ct}\left(I - e^{A_cT}\right)^{-1} \int_0^T e^{A_c(T-\tau)}\nu(\tau)d\tau + \int_0^t e^{A_c(t-\tau)}\nu(\tau)d\tau. \tag{16}$$

By using (16) in (3b) the existence of a periodic solution for (3) can be reformulated as the existence of a trajectory $z(t)$ solution of the following continuous-time linear complementarity problem, for any $t \in [0, T]$

$$0 \leqslant C_ce^{A_ct}\left(I - e^{A_cT}\right)^{-1} \int_0^T e^{A_c(T-\tau)}\nu(\tau)d\tau + \int_0^t C_ce^{A_c(t-\tau)}\nu(\tau)d\tau + D_cz(t) + F_cu(t) \perp z(t) \geqslant 0. \tag{17}$$

Finding a solution $z(t)$ for (17) is a not easy task. The alternative we propose consists in using a discretization approach.

B. Discretization of continuous LCP

Discretize the time interval $[0, T]$ in N time intervals of length θ:

$$0 < t_1 < \cdots t_k < \cdots < t_{N-1} < t_N = T,$$

with $t_k = k \cdot \theta$ and $T = N\theta$. By using(16) in (3b) evaluated in t_k

$$w(t_k) := w_k = C_c \left(e^{A_c\theta}\right)^k \left(I - \left(e^{A_c\theta}\right)^N\right)^{-1} \cdot \left[\sum_{i=1}^{N} \int_{(i-1)\theta}^{i\theta} e^{A_c(N\theta-\tau)}\nu(\tau)d\tau\right] + C_c \left[\sum_{i=1}^{k} \int_{(i-1)\theta}^{i\theta} e^{A_c(k\theta-\tau)}\nu(\tau)d\tau\right] + D_c z_k + F_c u_k. \tag{18}$$

By considering a backward Zero–Order–Hold discretization technique, we assume

$$\nu(\tau) = \nu_i \qquad \forall \tau \in (\,(i-1)\theta, i\theta\,] \tag{19}$$

for $i = 1, \ldots, N$. Then (18) becomes

$$w_k = C_c \left(e^{A_c\theta}\right)^k \left(I - \left(e^{A_c\theta}\right)^N\right)^{-1} \cdot \left[\sum_{i=1}^{N} \int_{(i-1)\theta}^{i\theta} e^{A_c(N\theta-\tau)}d\tau\, \nu_i\right] + C_c \left[\sum_{i=1}^{k} \int_{(i-1)\theta}^{i\theta} e^{A_c(k\theta-\tau)}d\tau\, \nu_i\right] + D_c z_k + F_c u_k \tag{20}$$

for $k = 1, \ldots, N$. By assuming A_c to be invertible we get

$$\begin{aligned}\int_{(i-1)\theta}^{i\theta} e^{A_c(k\theta-\tau)}d\tau &= \int_{(k-i)\theta}^{(k-i)\theta+\theta} e^{A_c y}dy \\ &= A_c^{-1} e^{A_c(k-i)\theta}\left(e^{A_c\theta} - I\right) \\ &= \left(e^{A_c\theta}\right)^{k-i} A_c^{-1}\left(e^{A_c\theta} - I\right).\end{aligned} \tag{21}$$

Define the following matrices:

$$A := e^{A_c\theta} \tag{22a}$$
$$B := A_c^{-1}\left(e^{A_c\theta} - I\right) B_c \tag{22b}$$
$$C := C_c A \tag{22c}$$
$$D := D_c + C_c B \tag{22d}$$
$$E := A_c^{-1}\left(e^{A_c\theta} - I\right) E_c \tag{22e}$$
$$F := F_c + C_c E \tag{22f}$$
$$\Pi := (I - A^N)^{-1}. \tag{22g}$$

Note that Π is well defined because we assume that A_c has nonzero eigenvalues. The matrix Π satisfies the following properties:

$$A\Pi = \Pi A \tag{23a}$$
$$\Pi A^N = \Pi - I. \tag{23b}$$

By using (23), with simple algebraic manipulations (20) can be rewritten as

$$w_k = CA^{k-1}\Pi\left[\sum_{i=1}^{N} A^{N-i}Bz_i + \sum_{i=1}^{N} A^{N-i}Eu_i\right] + C\left[\sum_{i=1}^{k-1} A^{k-1-i}Bz_i + \sum_{i=1}^{k-1} A^{k-1-i}Eu_i\right] + Dz_k + Fu_k \tag{24}$$

with $0 \leqslant w_k \perp z_k \geqslant 0$ for $k = 1, \ldots, N$. By using (23) the equations (24) can be rewritten as the following LCP(q,M):

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix} = q + M \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix} \tag{25a}$$

$$0 \leqslant w_k \perp z_k \geqslant 0, \quad k = 1, \ldots, N \tag{25b}$$

where

$$M = \begin{pmatrix} D & 0 & \cdots & 0 \\ 0 & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D \end{pmatrix} + \begin{pmatrix} C\Pi A^{N-1}B & C\Pi A^{N-2}B & \cdots & C\Pi B \\ C\Pi B & C\Pi A^{N-1}B & \cdots & C\Pi AB \\ \vdots & \vdots & \ddots & \vdots \\ C\Pi A^{N-2}B & C\Pi A^{N-3}B & \cdots & C\Pi A^{N-1}B \end{pmatrix} \tag{26}$$

and q is reported in (27). Note that M is a block circulant matrix. Moreover the LCP (25) cannot be decoupled into N different LCPs because for each k all components of the sequence z_i for $i = 1, \ldots, N$ appear into w_k.

C. Discretization of continuous LCS

The LCP (25)–(27) can be also obtained by discretizing the LCS (3). By discretizing (3a) with a backward Zero–Order–Hold technique with sampling period θ, it is possible to get the following discrete–time system:

$$x_k = Ax_{k-1} + Bz_k + Eu_k. \tag{28}$$

where the matrices A, B, E are given by (22). By substituting (28) in (3b) evaluated at step k, one obtains the following discrete–time linear complementarity system

$$x_k = Ax_{k-1} + Bz_k + Eu_k \tag{29a}$$
$$w_k = Cx_{k-1} + Dz_k + Fu_k \tag{29b}$$
$$0 \leqslant w_k \perp z_k \geqslant 0. \tag{29c}$$

Consider the system (29) forced by a periodic external signal $\{u_k\}$ of period N. Assume the system admits a periodic forced oscillation, i.e. $x_{k+N} = x_k\ \forall k$. The state evolution gives

$$x_N = A^N x_0 + \sum_{i=1}^{N} A^{N-i}\left(Bz_i + Eu_i\right) = x_0. \tag{30}$$

By solving with respect to x_0:

$$x_0 = \Pi \sum_{i=1}^{N} A^{N-i}\left(Bz_i + Eu_i\right). \tag{31}$$

Note that $u_i,\ i = 1, \ldots, N$ is a known input whereas $z_i,\ i = 1, \ldots, N$ are unknowns. By using (31) in (29b) for $k = 1, \ldots, N$ it is simple to show that the same LCP (25)–(27) is obtained again.

$$q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} C\Pi A^{N-1}E + F & C\Pi A^{N-2}E & \cdots & C\Pi E \\ C\Pi E & C\Pi A^{N-1}E + F & \cdots & C\Pi AE \\ \vdots & \vdots & \ddots & \vdots \\ C\Pi A^{N-2}E & C\Pi A^{N-3}E & \cdots & C\Pi A^{N-1}E + F \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \tag{27}$$

D. Continuous and discrete LCPs

Assume that the continuous–time complementarity system (3) has a periodic oscillation of period $T = \alpha T_s$ with α being an integer. Without loss of generality one can choose a sampling period θ such that $T_s = \beta\theta$, with β being a sufficiently large integer. Therefore it will be $T = \alpha T_s = \alpha\beta\theta = N\theta$, with $N = \alpha\beta$ being a sufficiently large integer. Note that the continuous–time instants at which conditions (3c) change, i.e. when one or more components of w or z become zero, do not need to be known a priori and do not need to be sampling time instants. In other words the sequence of modes of the periodic oscillation is not fixed a priori.

The LCP (25)–(27) might have no solution, one solution or multiple solutions. Each solution of the LCP (25)–(27) provides a periodic sequence z_k with $k = 1, \dots, N$, and, by using (29a)–(29b), one can also compute the corresponding sequence x_k. Therefore one would expect that each pair of periodic sequences (x_k, z_k) allows to compute an approximation of a periodic solution of (3). To this aim one should prove consistency of the discretization [27], [36], [37]. For instance one might prove that a continuous piecewise linear interpolant obtained from the periodic sequences (x_k, z_k) converges to a periodic solution of (3) when θ tends to zero. Such a consistency result is out of the scope of this paper and it is not easy to be proved. In [27] some contributions in this direction are proposed, but under assumptions which are not satisfied in our scenario, i.e. Lipschitz continuous inputs and excluding the boundary condition $x(T) = x(0)$.

In the sequel we assume that each solution of the discrete LCP (25)–(27) approximates a sample of a solution of the continuous LCP (17) and then it allows to compute a steady state periodic oscillation.

IV. Uniqueness of solution under passivity

The case when the LCP (25)–(27) has a unique solution is of particular interest. In this section we show that if the continuous–time model (3a)–(3b) is strictly passive with respect to the input z and the output w, then for a sufficiently small sampling period the discrete transfer function obtained from (29a)–(29b) with (22) is strictly positive real and the LCP (25)–(27) will have a unique solution. The relevance of such result is justified by the fact that for many power converters the passivity of a model in the form (3a)–(3b) can be simply stated by using circuit theory arguments [26].

A. Passivity and positive realness

Here we recall definitions of passivity and positive realness and their relationship.

***Definition** 4:* A continuous–time system (3a)–(3b) is said to be *passive* with respect to the input z and the output w if there exists a function $V : \mathbb{R}^n \to \mathbb{R}_+$ (called a storage function) such that

$$V(x(t_0)) + \int_{t_0}^{t_1} w^T(t)z(t)\,dt \geqslant V(x(t_1)) \tag{32}$$

holds for all $t_1 > t_0$, and for all solutions of (3a)–(3b) with $u = 0$.

A continuous–time system (3a)–(3b) is said to be *strictly passive* if the inequality (32) is strict.

The passivity can be related to the positive realness of the corresponding transfer function. From [38]:

***Definition** 5:* A transfer matrix $G_c(s)$ is said to be *strictly positive real* if:

- $G_c(s-\epsilon)$ has no pole in $Re[s] > 0$;
- $G_c(s-\epsilon)$ is real for all positive real s;
- $G_c(s-\epsilon) + G_c^*(s-\epsilon) \geqslant 0$ for all $Re[s] > 0$.

An important result that will be used below is that for linear–time–invariant systems the concepts of (strictly) passivity and (strictly) positive realness are equivalent.

A further useful definition is strictly positive realness for discrete–time systems.

***Definition** 6:* A proper rational discrete transfer matrix $G(z)$ is said to be *strictly positive real* if :

- poles of all elements of $G(z)$ are in $|z| < 1$;
- the matrix $G(e^{j\xi}) + G^T(e^{-j\xi})$ is positive definite for $\theta \in [-\pi, \pi)$.

B. Preserving positive realness under discretization

Consider (29a)–(29b) as being a discretization of (3a)–(3b) and assume that

$$G_c(s) = C_c(sI - A_c)^{-1}B_c + D_c \tag{33}$$

is strictly positive real. It is not obvious whether the discrete transfer matrix

$$G(z) = C(zI - A)^{-1}B + D \tag{34}$$

is strictly positive real or not. Indeed such implication depends on the particular discretization technique adopted. By using the backward Euler discretization technique the matrices in (29a)–(29b) are given by

$$A := (I - A_c\theta)^{-1} \tag{35a}$$
$$B := \theta\,(I - A_c\theta)^{-1}\,B_c \tag{35b}$$
$$C := C_c\,(I - A_c\theta)^{-1} \tag{35c}$$
$$D := D_c + \theta C_c\,(I - A_c\theta)^{-1}\,B_c. \tag{35d}$$

In [39] it has been proved that if (33) is positive real then (34) with (35) is discrete positive real as well. On the other hand by using the forward Euler discretization technique it can be easily found that positive realness is not preserved under discretization, for any sampling period. In what follows we show that by choosing a sufficiently small sampling period, positive realness is preserved under discretization with the backward Zero–Order–Hold technique, see (22).

***Theorem** 1:* Given a strictly positive real continuous transfer matrix (33), there exists a sufficiently small $\theta_0 > 0$ such that for any sampling period $\theta \leqslant \theta_0$ the discrete transfer matrix (34) obtained by using the backward Zero–Order–Hold discretization technique, i.e. by using (22a)–(22d), is strictly positive real.

Proof: See [40]. ■

C. Uniqueness of the LCP solution

We are now ready to show that if the transfer matrix (34) is strictly positive real, then the LCP (25) has a unique solution. To prove that we will use a frequency–domain analysis. Since the sequence z_k is periodic with $z_N = z_0$, its Discrete Fourier Transform (DFT) can be written as

$$Z(l) = \sum_{k=0}^{N-1} z_k \Omega^{kl} = \sum_{k=1}^{N} z_k \Omega^{kl} \tag{36}$$

with $\Omega = e^{-j2\pi/N}$. By applying the DFT to (29) we get:

$$\begin{aligned} W(l) &= \left[D + C\left(\Omega^{-l}I - A\right)^{-1} B\right] Z(l) \\ &\quad + \left[F + C\left(\Omega^{-l}I - A\right)^{-1} E\right] U(l). \end{aligned} \tag{37}$$

Since by definition the inverse DFT is

$$w_k = \frac{1}{N}\sum_{l=0}^{N-1} W(l)\Omega^{-kl} = \frac{1}{N}\sum_{l=1}^{N} W(l)\Omega^{-kl}, \tag{38}$$

then

$$\begin{aligned} w_k &= \frac{1}{N}\sum_{l=1}^{N}\left[\left(D + C\left(\Omega^{-l}I - A\right)^{-1} B\right)\left(\sum_{i=1}^{N} z_i\Omega^{il}\right)\right. \\ &\quad \left. + \left[F + C\left(\Omega^{-l}I - A\right)^{-1} E\right]\left(\sum_{i=1}^{N} u_i\Omega^{il}\right)\right]\Omega^{-kl} \\ &= \sum_{i=1}^{N}\left[\frac{1}{N}\sum_{l=1}^{N}\left(D + C\left(\Omega^{-l}I - A\right)^{-1} B\right)\Omega^{(i-k)l}\right] z_i \\ &\quad + \sum_{i=1}^{N}\left[\frac{1}{N}\sum_{l=1}^{N}\left(F + C\left(\Omega^{-l}I - A\right)^{-1} E\right)\Omega^{(i-k)l}\right] u_i \end{aligned} \tag{39}$$

for $k = 0, \ldots, N-1$. Since for periodicity $u_0 = u_N$, $z_0 = z_N$, $w_0 = w_N$, by exploiting the circulant properties of the matrices the problem can be more compactly rewritten as

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix} = \tilde{q} + \tilde{M}\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix} \tag{40a}$$

$$0 \leqslant w_k \perp z_k \geqslant 0, \quad k = 1, \ldots, N. \tag{40b}$$

where $\tilde{M}$ is a block matrix circulant whose block k-th, i-th is

$$\{\tilde{M}\}_{k,i} = \frac{1}{N}\sum_{l=1}^{N}\left(D + C\left(\Omega^{-l}I - A\right)^{-1} B\right)\Omega^{(i-k)l}. \tag{41}$$

The expression (41) can be interpreted as the inverse DFT of the frequency response of the cyclic linear–time–invariant system $(A,\, B,\, C,\, D)$ [41]. By construction it follows that (40) is equal to (25) and then $q = \tilde{q}$ and $M = \tilde{M}$. So as mentioned above, by using the formulation (25) it is not easy to show the uniqueness of the LCP solution. We will tackle such issue by using (40), instead.

In particular we can prove the following

***Theorem** 2:* Assume the discrete transfer matrix 34 is strictly positive real. Then the LCP (40)–(41), or equivalently the LCP (25)–(27), has a unique solution.

Proof: See [40]. ■

V. Simulation results

In this section we apply the proposed algorithm to a dc/dc buck converter and to a resonant converter. By considering open loop operating conditions for the (passive) dc/dc buck converter the accuracy of the proposed approach for the computation of the control–to–output frequency response, both in continuous and discontinuous conduction modes, will be shown. Then, the buck converter under closed loop voltage mode control will be analyzed as an example of non passive circuit and the corresponding multiple periodic steady state solutions will be computed. Finally a resonant converter example for which the computation of steady state behavior is of practical importance will be presented. In all scenarios the different LCP problems will be solved by using the very efficient PATH algorithm [42] and, where needed, by implementing the algorithm proposed in [43] for finding multiple solutions.

A. DC-DC buck converter

Consider the dc-dc buck converter depicted in Fig. 4 with open loop pulse width modulation, i.e. v_c in (7) is an exogenous constant or periodic signal and not given by (9). The LCP (25)–(27) can be used to compute the frequency control–to–output response of the converter. To this aim, in the particular case when the converter operates in continuous conduction mode, one can estimate the accuracy of the complementarity solution by comparing the results with those achievable by using the averaged model. Since we are assuming continuous conduction mode, the averaged model provides [32], [33]:

$$\tilde{X}_2(s) = G(s)\,\frac{V_{dc}}{\Delta} V_c(s), \tag{42}$$

with $\tilde{x}_2$ being the average values of x_2 and

$$G(s) = \frac{\dfrac{R_2}{R_1 + R_2}}{\dfrac{L_1 C_2 R_2}{R_1 + R_2}s^2 + \dfrac{C_2 R_1 R_2 + L_1}{R_1 + R_2}s + 1}. \tag{43}$$

For this particular case one can also compute the Laplace transform of the output of the circuit in Fig. 4 with the diode replaced with a short circuit (the converter is assumed to operate in continuous conduction mode) and v_{in} being a square wave signal, see (7).

$$X_2(s) = G(s)\, V_{in}(s). \tag{44}$$

First we assume a constant control signal $v_c = V_0$. As it is well known the typical steady state behavior is a periodic oscillation with period T_s. We want to use the LCP approach for computing the average value of the output x_2 under such conditions. In this case, of course, the averaged model (42) provides correctly the same value obtained through the analytical model (44) since we are evaluating the values at zero frequency. Thus we can define the relative percentage error given by the LCP solution as

$$\left| \frac{1}{N} \sum_{k=1}^{N} x_{2k} - G(0) \frac{V_{dc}}{\Delta} V_0 \right| \frac{\Delta}{G(0) V_{dc} V_0} \cdot 100, \tag{45}$$

where x_{2k} is the output voltage computed with the LCP approach and N the number of samples per period such that $h = T_s/N$.

If the LCP solution have been affected only by a discretization error due to the fact that the switching time instant is evaluated as the discrete time instant closest to a multiple of the sampling period, the expression (45) would be (it is not difficult to prove that)

$$\frac{\Delta}{V_0 N} \left| 1 + \left\lfloor \frac{V_0 N}{\Delta} \right\rfloor - \frac{V_0 N}{\Delta} \right| \cdot 100, \tag{46}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Consider the following converter parameters: $R_1 = 0.1\,\Omega$, $R_2 = 12.5\,\Omega$, $L = 2.08\,\text{mH}$, $C_2 = 100\,\text{nF}$, $V_{dc} = 33\,\text{V}$, $\Delta = 1\,\text{V}$, $T_s = 1/30\,\text{ms}$, and the control voltage as a constant $v_c = V_0 = 0.3\,\text{V}$. In this scenario numerical simulations show that the error introduced by the LCP method is almost completely due to the discretization. Indeed in Fig. 5 a histogram analysis on the relative percentage error due to the LCP method is reported and it shows that the proposed approach in this case introduces only numerical errors that can be assumed negligible. Since v_{in} is a squarewave of known period and amplitude, by using the frequency response corresponding to (44) one can obtain the values also for the first and higher order harmonics of the output voltage. In particular, confirmed by the results of the LCP (24), one can easily compute an average output voltage $9.8309732\,\text{V}$, the first harmonic (at $f_s = 1/T_s = 30\,\text{kHz}$) with $0.3095763\,\text{V}$ amplitude and zero phase, and the second harmonic (at $2f_s = 2/T_s = 60\,\text{kHz}$) with $0.1437689\,\text{V}$ amplitude and zero phase.

A typical approach in order to obtain the control–to–output frequency response of power converters consists of assuming a control signal to be sinusoidal with nonzero mean: $v_c(t) = V_0 + V_1 \sin(2\pi t/T_c)$. Without loss of generality it is useful to consider a control voltage period proportional to the carrier signal period, i.e. $T_c = N_c T_s$ with N_c positive integer. In order to fix the size N of the LCP problem to be solved, the sampling period is chosen such that $h = T_c/N = N_c/NT_s$. Fig. 6 shows the results obtained for the first harmonic by varying $N_c = 2, 3, \ldots, 10$ with $V_1 = 0.03\,\text{V}$ and $N = 343$. The figure confirms the good accuracy of the LCP solution and the well known result that the accuracy of the averaged model (42) decreases when the ratio between the carrier frequency and the modulating signal frequency decreases.

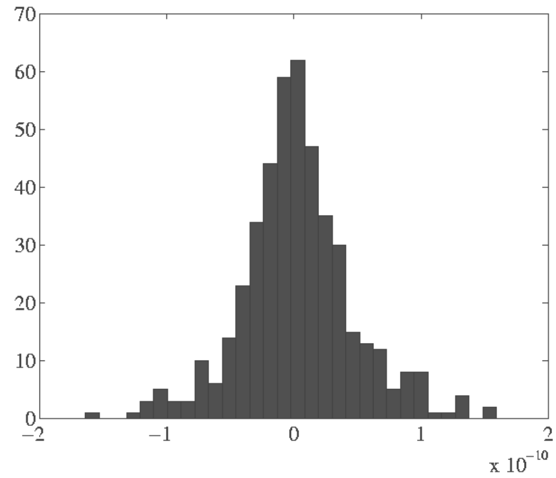

Fig. 5. Histogram of the difference between the values given by (45) and the values given by (46), for $N = 13, 16, 19, \ldots, 1357$.

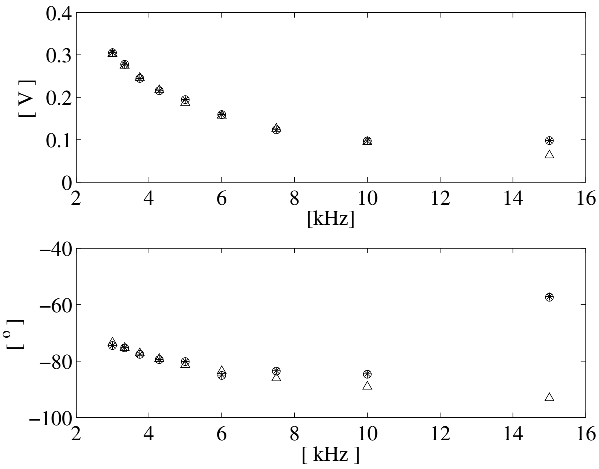

Fig. 6. First harmonics of the output voltage obtained with the LCP solution ($*$), the averaged model ($\triangle$), and the frequency response corresponding to (44) ($\circ$), for different values of N_c.

The results presented above allow to confirm the good accuracy obtained by using the LCP approach for computing the cyclic steady state behavior of the converter. The LCP approach becomes much more interesting when the Laplace transform (44) cannot be computed because, for instance, the converter operates in discontinuous conduction mode. In this case Fig. 7 shows the results obtained for the first harmonic of the LCP solution by varying $N_c = 2, 3, \ldots, 10$ with $R_2 = 250\,\Omega$. Note that in this case the Bode diagram of $G(s)$ presents a resonance peak at 10 kHz.

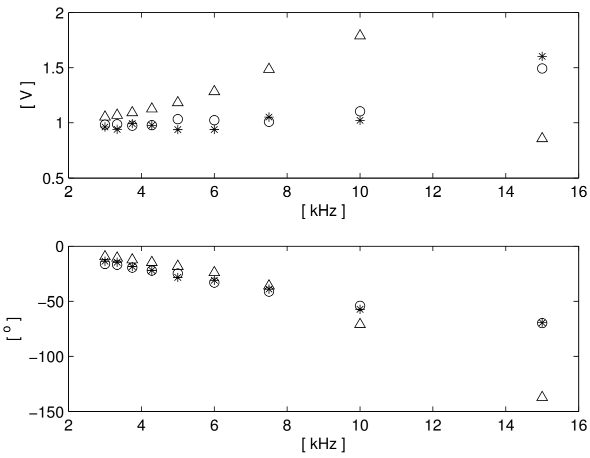

Fig. 7. First harmonics of the output voltage obtained with the LCP solution (∗), with the “wrong” averaged model (△), and by evaluating the steady–state of time–stepping simulations with PLECS [44] (○), under discontinuous conduction mode, for different values of N_c.

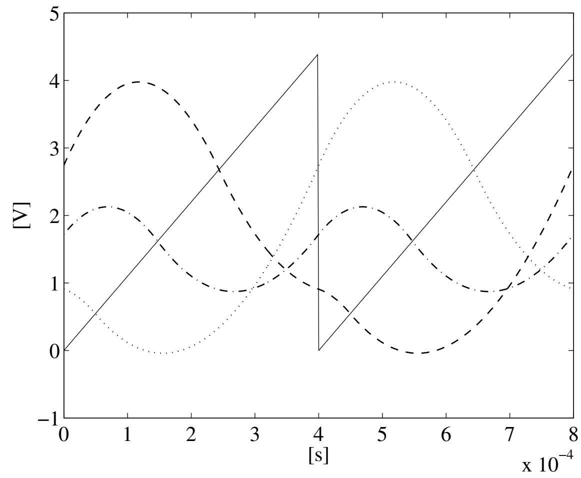

Fig. 8. Carrier signal δ (continuous line) and control voltages v_c for stable solutions with period $2\,T_s$ (dashed and dotted lines) and unstable solution of period T_s (dash–dotted line).

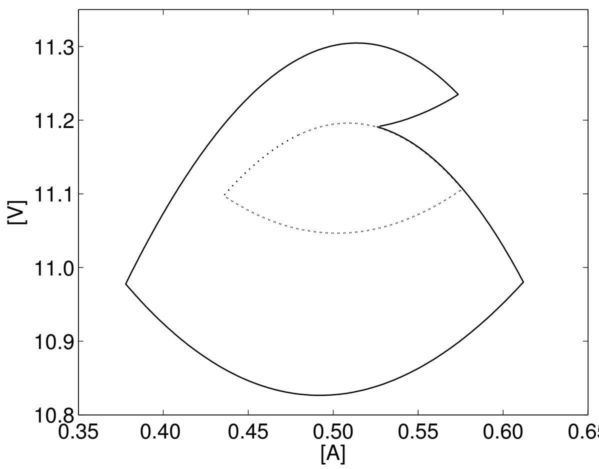

Fig. 9. Phase plane corresponding to solutions in Fig. 8: unstable (dotted) and stable (continuous).

B. Closed loop buck converter with multiple solutions

Consider the dc/dc buck converter shown in Fig. 3 with the following parameters: $V_{dc} = 30\,\text{V}$, $R_1 = 0\,\Omega$, $L_1 = 20\,\text{mH}$, $C_2 = 47\,\mu\text{F}$, $R_2 = 22\,\Omega$, $k_p = 8.4$, $T_s = 400\,\mu\text{s}$, $\Delta = 4.4$, $V_{\text{ref}} = 11.3\,\text{V}$. As shown in [45] this circuit exhibits a period doubling bifurcation, under the variation of the input voltage V_{dc}. Indeed, for $V_{dc} \in [15, 24]\,\text{V}$ we have one stable periodic solution with period T_s, whereas for $V_{dc} \in (24, 32]\,\text{V}$ the closed loop system exhibits an unstable periodic solution of period T_s and a stable periodic solution of period $2T_s$. This is confirmed by the fact that for $V_{dc} = 30\,\text{V}$ the continuous–time model (3a)–(3b) is not passive with respect to the input z and the output w, hence we might have no solution, one solution or multiple solutions for the LCP (25)–(27). In particular, for $V_{dc} = 30\,\text{V}$ we expect to find at least two solutions for the LCP (25)–(27): the stable one with period $2\,T_s$, and the unstable one with period T_s.

Using a sampling period of $1.6\,\mu\text{s}$ we have 250 points within T_s and we look for 500–points solutions. By applying the algorithm presented in [43], it is possible to find three solutions: the unstable solution with period T_s and two stable solutions with period $2\,T_s$, see Fig. 8. In particular, we get two stable solutions with period $2\,T_s$ because the carrier signal δ, which is an input for the LCP (25)–(27), is periodic with period T_s. Therefore one solution can be obtained from the other by means of a time shift of T_s. Fig. 9 shows the different solutions in the phase plane. The two stable solutions obviously overlap in the phase plane.

Note that the steady state solutions are obtained without knowing apriori the sequence of modes nor applying the Poincaré maps. In particular, the unstable steady state solutions is hard to find with methods based on time–stepping simulations.

C. Steady state oscillation of resonant converters

Consider the dc/dc LLC resonant converter displayed in Fig. 2 and assume the following circuit parameters: $V_{dc} = 42\,\text{V}$, $R_1 = 200\,\text{m}\Omega$, $C_1 = 138\,\text{nF}$, $L_1 = 7.6\,\mu\text{H}$, $n = 1.64$, $C_2 = 100\,\mu\text{F}$. Define the following circuit parameters:

$$\omega_0 = \frac{1}{\sqrt{L_1 C_1}},\ \rho = \frac{2\pi f_s}{\omega_0},\ Q = \frac{\omega_0 L_1}{n^2 R_2} \qquad (47a)$$

$$A_L = \frac{L_2}{L_1},\ M_{\text{out}} = \frac{n V_{\text{out}}}{V_{dc}} \qquad (47b)$$

where V_{out} is the average value over the period T of the output voltage x_4 at steady state. In Table I there are reported the values of M_{out} obtained for $Q = 0.1$ and $A_L = 1$ by solving the LCP problem described in the previous section and by using time–stepping simulations with PLECS [44], for two values of the parameter ρ. For each value of ρ

N	LCP $\rho = 1.00$	PLECS $\rho = 1.00$	LCP $\rho = 0.723$	PLECS $\rho = 0.723$
100	0.51223	0.51166	3.7796	3.7125
200	0.51204	0.51189	3.7784	3.7685
300	0.51200	0.51194	3.7777	3.7705
400	0.51199	0.51195	3.7775	3.7743
500	0.51198	0.51196	3.7775	3.7754
600	0.51198	0.51196	3.7774	3.7758
700	0.51197	0.51197	3.7774	3.7765

TABLE I
CONVERGENCE RATE FOR THE PARAMETER M_{out} IN THE SOLUTION OF THE COMPLEMENTARITY PROBLEM WITH RESPECT TO THE PLECS FIXED TIME–STEPPING SIMULATIONS.

the solutions obtained with different number of samples N per period $T = 1/f_s$ (which is fixed by choosing ρ) are shown. The results obtained by using the LCP and PLECS are coherent, in the sense that the fixed time–step chosen in PLECS is $\theta = T/N$. For each value of ρ, since T is fixed, the accuracy of the solutions increases with N because a larger number of sampling time instants per period are chosen. It is clear the validity of the solutions obtained with the LCP that reproduce the results obtained with the PLECS simulations. The ratio between the average time needed to obtain the steady state with the PLECS time–stepping simulations and the time needed for solving the corresponding LCP increases with N and goes approximatively from 1 to 10.

Fig. 10 and Fig. 11 display the steady state results obtained by solving the corresponding LCPs for different values of the load resistance, with $N = 300$ and A_L corresponding to $L_2 = 9.7\,\mu$H and $L_2 = L_1$, respectively.

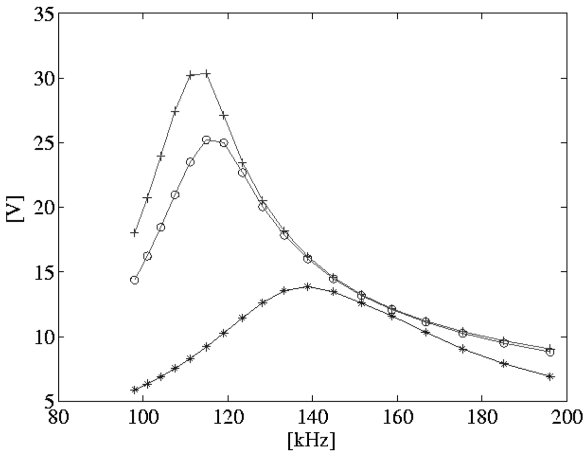

Fig. 10. Average output voltage V_{out} vs. the switching frequency f_s ($\rho \in [6, 12] \cdot 10^{-4}$), for different values of the load resistances: $R_2 = 6.0\,\Omega$ ('+'-points), $R_2 = 4.5\,\Omega$ ('o'-points), $R_2 = 1.5\,\Omega$ ('$*$'-points). The continuous lines are the interpolations of the LCP solutions and the points are the PLECS results.

VI. CONCLUSION

Complementarity formalism has been recently proposed as a framework for representing complete switched model of power converters and for their time–domain analysis. The complementarity model can be constructed without explicitly detailing all modes of the converter and without a priori knowledge of the sequence of modes in the converter dynamic evolution. In this paper it has been shown how the complementarity framework can be used to compute periodic oscillations for power converters. Conditions for the existence of periodic oscillations in terms of solvability of a suitable static linear complementarity problem are obtained. The proposed procedure does not need to fix a priori the shape of the periodic oscillation and the examples considered have shown how it is possible to capture the steady state behavior also for complex scenarios such as power converters that exploit resonances, operating in discontinuous conduction mode and exhibiting unstable orbits.

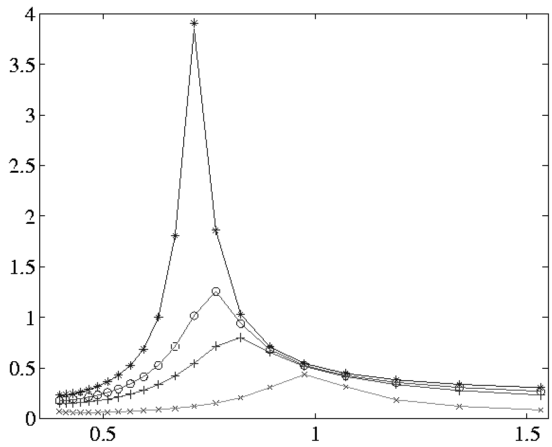

Fig. 11. Converter gain M_{out} vs. ρ, for different values of the quality factor: $Q = 0.1$ ('$*$'-points), $Q = 0.5$ ('o'-points), $Q = 1$ ('+'-points), $Q = 5$ ('x'-points).

Furthermore, as an instrumental result aimed to prove uniqueness of the complementarity problem solution, in this paper it has been shown how the positive realness of the continuous time transfer matrix is preserved under the backward ZOH discretization technique, for sufficiently small discretization step, thus partly extending what already known in the literature regarding the backward Euler discretization.

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APPENDIX A
MATRICES OF THE CONVERTERS MODELS

The matrices of the complementarity model (3) of the resonant converter, obtained from (6), are

$$A_c = \begin{pmatrix} -R_1/L_1 & -1/L_1 & 0 & -n/L_1 \\ 1/C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & n/L_2 \\ n/C_2 & 0 & -n/C_2 & -1/(R_2C_2) \end{pmatrix}$$

$$B_c = \begin{pmatrix} n/L_1 & 0 \\ 0 & 0 \\ -n/L_2 & 0 \\ 0 & 2/C_2 \end{pmatrix}, \; E_c = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$C_c = \begin{pmatrix} n & 0 & -n & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \; D_c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; F_c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The matrices of the complementarity model (3) of the voltage–mode controlled buck converter, obtained by elaborating (10)-(8), are

$$A_c = \begin{pmatrix} -\frac{R_1}{L_1} & -\frac{1}{L_1} \\ \frac{1}{C_2} & -\frac{1}{R_2 C_2} \end{pmatrix}, \; B_c = \begin{pmatrix} \frac{1}{L_1} & -\frac{V_{dc}}{L_1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_c = \begin{pmatrix} V_{dc}/L_1 & 0 \\ 0 & 0 \end{pmatrix}, \; C_c = \begin{pmatrix} 1 & 0 \\ 0 & -k_p \\ 0 & 0 \end{pmatrix}$$

$$D_c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -V_{dc} & 0 \end{pmatrix}, \; F_c = \begin{pmatrix} 0 & 0 \\ k_p V_{ref} & -1 \\ V_{dc} & 0 \end{pmatrix}$$

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