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Averaging of Nonsmooth Systems using Dither

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Abstract

It was shown by Zames and Shneydor and later by Mossaheb that a high-frequency dither signal of a quite arbitrary shape can be used to narrow the effective nonlinear sector of Lipschitz continuous feedback systems. In this paper it is shown that also discontinuous nonlinearities of feedback systems can be narrowed using dither, as long as the amplitude distribution function of the dither is absolutely continuous and has bounded derivative. The averaged system is proven to approximate the dithered system with an error of order of the dither period.

Key words: Averaging theory; discontinuous control; dither; hybrid systems; switched systems; nonsmooth systems.

1 Introduction

A frequently used technique to stabilize a nonlinear feedback system in Luré form is by injecting a high-frequency dither signal, which narrows the nonlinear sector. If the dither frequency is sufficiently high, the behavior of the dithered system will be qualitatively the same as an averaged system, whose nonlinearity is the convolution of the amplitude distribution of the dither and the original nonlinearity. Analysis and control design can then be carried out on the averaged system, which in most cases is simpler to analyze due to lack of external dither signal and narrower nonlinearity. For the case when the original nonlinearity is Lipschitz continuous, the scheme outlined above was rigorously justified using properties of the amplitude distribution function of the dither [23,24]. Similar results were obtained later using classical averaging theory [12].

The Lipschitz continuity assumption on the nonlinearity of the dithered system is often violated in practice. Indeed, discontinuous nonlinearities in feedback systems with high-frequency excitations appear in a large variety of applications, including systems with adaptive control [2], friction [1], power electronics [11], pulse-width modulation [13], quantization [6], relays [20], and variable-structure control [21]. It is common to analyze these systems using empirical methods such as describing functions, which can give a quite good intuitive understanding. It is hard, however, to get bounds on the approximation these methods provide and they may even give erroneous results, so therefore there is a need for a solid treatment of discontinuous systems with high-frequency excitation. Recently, certain classes of these systems have been thoroughly studied, such as power

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converters [11], pulse-width modulated systems [5,18], relay systems [8], and stick-slip drives [16].

The main contribution of the paper is an averaging theorem for a general class of nonsmooth systems with a quite arbitrary periodic dither. The result states that the dithered and the averaged systems have qualitatively the same behavior when the dither has sufficiently high frequency and an absolutely continuous amplitude distribution function with bounded derivative. The averaging theorem might be interpreted as an extension to nonsmooth feedback system of previous results, which were limited to Lipschitz-continuous systems [23,24,12].

The outline of the paper is as follows. The dithered system and the corresponding averaged system are introduced in Section 2. The amplitude distribution function of the dither signal is thoroughly discussed, since it plays a key role in the analysis. The main result on the approximation error between the dithered and the averaged systems is presented in Section 3. The paper is concluded in Section 4 and the proofs are reported in Appendix.

2 Preliminaries

2.1 Dithered System

The dithered feedback system is defined as

$$\dot{x}(t) = f_0(x(t),t) + \sum_{i=1}^{m} f_i(x(t),t) n_i(g_i(x(t),t) + \delta_i(t)),$$
$$x(0) = x_0. \tag{1}$$

The state x belongs to $\mathbb{R}^q$. The functions $f_i : \mathbb{R}^q \times \mathbb{R} \to \mathbb{R}^q$, $i = 1, \ldots, m$, are assumed to be globally Lipschitz with respect to both x and t, i.e., there exists a positive constant L_f such that for all $x_1, x_2 \in \mathbb{R}^q$ and $t_1, t_2 \geq 0$,

$$|f_i(x_1,t_1) - f_i(x_2,t_2)| \leq L_f \left(|x_1 - x_2| + |t_1 - t_2|\right).$$

We further assume that f_0 is piecewise continuous with respect to t, $f_0(0,t) = 0$ for all $t \geq 0$, and

$$|f_0(x_1,t) - f_0(x_2,t)| \leq L_f \, |x_1 - x_2|$$

for all $x_1, x_2 \in \mathbb{R}^q$ and $t \geq 0$. Similarly, the functions $g_i : \mathbb{R}^q \times \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, m$, are assumed to have a common Lipschitz constant $L_g > 0$, i.e.,

$$|g_i(x_1,t_1) - g_i(x_2,t_2)| \leq L_g \left(|x_1 - x_2| + |t_1 - t_2|\right)$$

for all $x_1, x_2 \in \mathbb{R}^q$, $t_1, t_2 \geq 0$. The nonlinearities $n_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, m$, are assumed to be functions of bounded variation. Recall that the total variation TV of a function $n : \mathbb{R} \to \mathbb{R}$ is

$$TV(n) \triangleq \sup_{-\infty < z_0 \leq z_1 \leq \cdots \leq z_k < \infty} \sum_{i=1}^{k} |n(z_i) - n(z_{i-1})|,$$

where the supremum is taken over all finite sequences $\{z_i\}_{i=0}^{k}$ with $k \geq 1$ [22]. If the total variation is bounded, we simply say that n is of bounded variation. Hence, the functions n_i can be discontinuous, but they are necessarily bounded. Each dither signal $\delta_i : [0,\infty) \to \mathbb{R}$ is supposed to be a p-periodic measurable function bounded by a positive constant M_δ, i.e., $|\delta_i| \leq M_\delta \; \forall i$.

When the differential equation (1) has a discontinuous right-hand side (due to that at least one n_i is discontinuous), existence and uniqueness of solutions depend critically on the considered definition of solution [4]. In the following we assume that the differential equation (1) has at least one absolutely continuous solution $x(t, x_0)$ on $[0,\infty)$ (in the sense of Carathéodory). We suppose that the time intervals when the solution is at a discontinuity point of n_i are of zero Lebesgue measure. Note that as a consequence, we do not consider solutions with sliding modes. Furthermore we suppose that the solutions have no accumulation of switching events (Zeno solutions).

The assumptions on system (1) imply that there exists a positive constant L_x such that $|x(t_1) - x(t_2)| \leq L_x|t_1 - t_2|$ for almost all $0 \leq t_1 \leq t_2 < \infty$. Estimates of the Lipschitz constant L_x can be easily obtained on any compact interval.

Remark 1 *The assumption on the nonlinearity n_i is weak. The class of considered systems thus contains quite exotic differential equations for which, for example, existence and uniqueness of solution cannot easily be addressed. However, for most cases in applications the existence of a Carathéodory solution is reasonable. Existence and uniqueness of solutions for dithered relay systems are discussed in [10].*

Remark 2 *The assumption on global Lipschitz continuity of the functions f_i, g_i is used to derive the Lipschitz bound L_x. The assumption can be relaxed by assuming Lipschitzness on a bounded set provided that dithered and averaged solutions belong to such set., see [19].*

2.2 Dither Signals and Their Amplitude Distribution Functions

Definition 2.1 *The amplitude distribution function $F_\delta : \mathbb{R} \to [0,1]$ of a p-periodic dither signal $\delta : [0,\infty) \to \mathbb{R}$ is defined as*

$$F_\delta(\xi) \triangleq \frac{1}{p}\mu\left(\{t \in [0,p) : \delta(t) \leq \xi\}\right)$$

where μ denotes the Lebesgue measure.

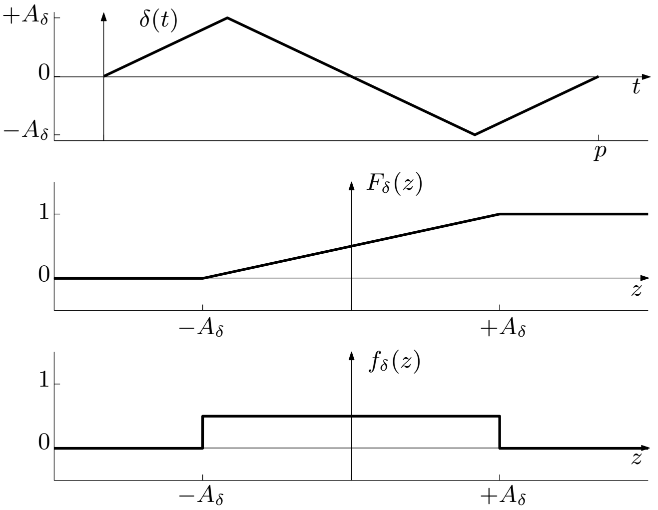

Fig. 1. Triangular dither signal with its corresponding amplitude distribution function and amplitude density function.

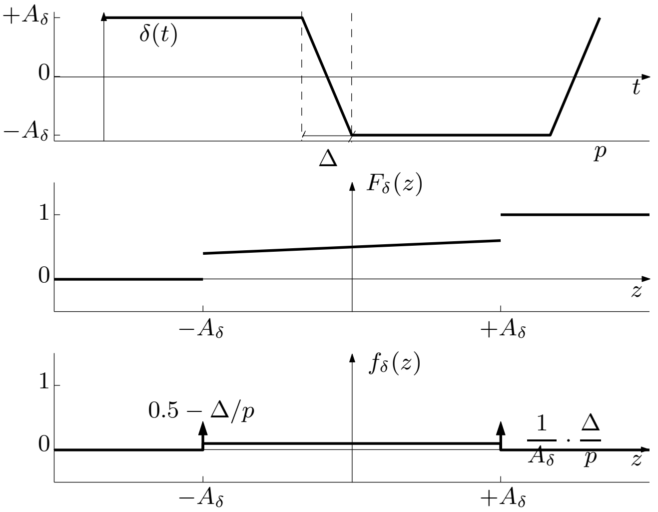

Fig. 2. Trapezoidal dither with its amplitude distribution function and the corresponding generalized derivative.

When the amplitude distribution function is absolutely continuous (with respect to its Lebesgue measure), the amplitude density function $f_\delta(\xi)$ is defined as

$$f_\delta(\xi) \triangleq \frac{\mathrm{d}F_\delta}{\mathrm{d}\xi}(\xi)$$

which exists almost everywhere.

The amplitude density and amplitude distribution functions play in a deterministic framework the same role as the probability density and cumulative distribution functions play in a stochastic framework. In particular, the amplitude distribution function is bounded, monotonously increasing, continuous from the right, and, if it is absolutely continuous, its derivative corresponds to the amplitude density function.

Typical dither signals are sawtooth, triangular, sinusoidal, trapezoidal, and square wave signals. Fig. 1 shows a triangular dither signal together with its amplitude distribution function F_δ and amplitude density function f_δ. A sawtooth dither with amplitude A_δ and period p has the same amplitude distribution function. For a trapezoidal dither, the amplitude distribution and its generalized derivative are reported in Fig. 2. Note that square wave dither corresponds to $\Delta = 0$. It is easy to see that dither signals that are constant over nonvanishing time intervals, such as trapezoidal and square wave signals, have discontinuous amplitude distribution function, which is thus in contrast to triangular dither.

2.3 Averaged System

The averaged system is given by

$$\begin{aligned} \dot{w}(t) &= f_0(w(t),t) + \sum_{i=1}^{m} f_i(w(t),t) N_i(g_i(w(t),t)), \\ w(0) &= w_0, \end{aligned} \tag{2}$$

where N_i is the averaged nonlinearity defined as follows.

Definition 2.2 *For each dither signal $\delta : [0,\infty) \to \mathbb{R}$ and nonlinearity $n : \mathbb{R} \to \mathbb{R}$ the averaged nonlinearity $N : \mathbb{R} \to \mathbb{R}$ is defined as*

$$N(z) \triangleq \int_{\mathbb{R}} n(z+\xi)\,\mathrm{d}F_\delta(\xi) \tag{3}$$

where the integral is a Lebesgue–Stieltjes integral.

In many cases the averaged nonlinearity can be formulated as a time average, as the following lemma states.

Lemma 2.1 *[3,17] The following equality holds provided that either side exists:*

$$\int_{\mathbb{R}} n(z+\xi)\,\mathrm{d}F_\delta(\xi) = \frac{1}{p}\int_{[0,p)} n(z+\delta(s))\,\mathrm{d}s.$$

It is interesting to investigate some aspects related to the continuity of the amplitude distribution function. When the amplitude distribution function is absolutely continuous, we have

$$N(z) = \int_{\mathbb{R}} n(z+\xi)\,\mathrm{d}F_\delta(\xi) = \int_{\mathbb{R}} n(z+\xi) f_\delta(\xi)\,\mathrm{d}\xi,$$

which is well defined under the given assumptions on n.

When the Lebesgue–Stieltjes measure corresponding to the amplitude distribution function has a decomposition (relative to the Lebesgue measure) into an absolutely continuous part with derivative f_δ and an atomic part, we have

$$N(z) = \int_{\mathbb{R}} n(z+\xi)\,\mathrm{d}F_\delta(\xi)$$
$$= \int_{\mathbb{R}} n(z+\xi) f_\delta(\xi)\,\mathrm{d}\xi + \sum_k n(z+\xi_k)F_k, \quad (4)$$

where $F_k \neq 0$ are the jump discontinuities corresponding to the atomic parts of the measure defined by the amplitude distribution function. Square wave and trapezoidal dither signals have this kind of amplitude distribution functions, cf., Fig. 2. Equation (4) is well defined except at possible discontinuity points of n. Thus in the case in which n is continuous, the results of Zames and Shneydor [23] and Mossaheb [12] can be applied together with equation (4) to compute the averaged system. If the amplitude distribution function is discontinuous in ξ_k and n is discontinuous in $z+\xi_k$, then neither these results from the literature nor the averaging theorem in Section 3 can be applied. Indeed, it can be shown that the averaged and the dithered systems can behave qualitatively quite different when n is discontinuous. See [10] for an illustrative example of this case.

3 Averaging Theorem

The main result of the paper states conditions under which the averaged system approximates the behavior of the dithered system for a sufficiently high dither frequency.

Theorem 3.1 *Consider the dithered system* (1) *and the averaged system* (2) *under the following assumptions:*

- (*i*) *the dithered system has an absolutely continuous solution,*
- (*ii*) *f_i and g_i are globally Lipschitz with Lipschitz constants L_f and L_g, respectively,*
- (*iii*) *f_0 is globally Lipschitz with respect to x with Lipschitz constant L_f, and $f_0(0,t)=0$,*
- (*iv*) *n_i is a function of bounded variation,*
- (*v*) *each dither δ_i is p-periodic, $|\delta_i| \leq M_\delta$, and has absolutely continuous amplitude distribution function F_{δ_i} with derivative bounded by $L_F \triangleq \max_i \sup_{\xi\in\mathbb{R}} |f_{\delta_i}(\xi)| < \infty$.*

Then, the averaged nonlinearities N_i are globally Lipschitz continuous and the averaged system (2) has a unique absolutely continuous solution on $[0,\infty)$. *Moreover, for any compact set $\mathcal{K} \subset \mathbb{R}^n$ and any $T>0$, there exists a positive constant $c(\mathcal{K},T)$ such that*

$$|x(t,x_0) - w(t,x_0)| \leq c(\mathcal{K},T)p, \quad \forall x_0 \in \mathcal{K},\ t\in[0,T] \quad (5)$$

PROOF. See Appendix.

Remark 3 *It is possible to relax the periodicity assumption on the dither and instead consider F-repetitive dither signals as in [23]. A dither signal δ is F-repetitive if there exists an unbounded sequence $\{t_k\}$, $0=t_0<t_1\ldots$, of sampling times such that (1) the maximal repetition interval $\sup_k(t_k-t_{k-1})$ is bounded and (2) the amplitude distribution function of δ on every interval (t_{k-1},t_k) is equal to the amplitude distribution function of δ on (t_0,t_1). See [10] for details.*

Remark 4 *The statement of the theorem appears to be fairly tight, because examples suggest that dithering might loose its effect when the hypotheses are violated. In particular, the dithered and the averaged solution may have qualitatively different behavior when the averaged nonlinearity is not Lipschitz continuous. Experimental confirmation of such behaviors on a DC motor are provided in [9]. We have discovered similar phenomena for limit cycles of the averaged and the dithered systems in [8,7,9]. The reason for the different behaviors in these examples is that averaged solution converges to a point of discontinuity of the nonlinearity, while the dithered system has a solution with a small amplitude ripple that perturb the solution across the boundary of the discontinuity. This behaviour gives rise to a new type of oscillation of the dithered system, which deserves more careful analysis. Obviously, a bound as in (5), which is uniform in every given compact set, cannot be fulfilled when the qualitative presence of the dithered and the averaged systems are so different.*

Remark 5 *Dithering can be interpreted as a technique for regularization of solutions of nonsmooth systems. In fact, if n is discontinuous, the solution of* (1) *might not be unique. On the other hand, if the amplitude distribution function of the dither is Lipschitz, then the averaged nonlinearity will be Lipschitz, so the averaged system* (2) *will have a unique solution. Now, from Theorem 3.1 one can conclude that by decreasing the dither period, all possible solutions of* (1) *will become closer and closer to the unique solution of the averaged system* (2).

4 Conclusions

It was shown that a high-frequency dither signal of a quite arbitrary shape can be used to narrow the effective nonlinear sector of nonsmooth feedback systems. The result can be interpreted as an extension of existing results for Lipschitz-continuous systems. The main theorem related the dynamics of the dithered system with an averaged system and stated that the approximation error is of the order of the dither period, under the condition that the amplitude distribution function of the dither is absolutely continuous and has bounded derivative.

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A Proof of Theorem 3.1

The proof is based on three lemmas. In the first lemma we show that the averaged nonlinearity under our assumptions is Lipschitz continuous, which implies that there exists a unique absolutely continuous solution of the averaged system on any finite time-horizon.

Lemma A.1 *Suppose n is of bounded variation with total variation $TV(n)$ and that F_δ is absolutely continuous with derivative f_δ and $L_F = \sup_{\xi\in\mathbb{R}} |f_\delta(\xi)| < \infty$. Then $\|n\|_\infty \le M_n$ and*

$$N(z) = \int_{\mathbb{R}} n(z+\xi) f_\delta(\xi)\,\mathrm{d}\xi$$

has a Lipschitz constant $L_N \le L_F TV(n)$ and $\|N\|_\infty \le \|n\|_\infty$.

PROOF. Since n is of bounded variation, it follows that $\|n\|_\infty \le M_n$, for some $M_n > 0$. Moreover we have

$$\begin{aligned}|N(z_1) - N(z_2)| &= \left|\int_{\mathbb{R}} [n(z_1+\xi) - n(z_2+\xi)]\,\mathrm{d}F_\delta(\xi)\right| \\ &= \left|\int_{\mathbb{R}} n(\xi)\,[\,\mathrm{d}F_\delta(\xi - z_1) - \,\mathrm{d}F_\delta(\xi - z_2)]\right|\end{aligned}$$

Let $V(\xi) = F_\delta(\xi - z_1) - F_\delta(\xi - z_2)$. We have $V(\xi) = 0$ for $\xi \notin S = [-M_\delta + \min(z_1, z_2), M_\delta + \max(z_1, z_2)]$. Hence, for any $I = [a, b] \supset S$ integration by parts gives

$$\begin{aligned}|N(z_1) - N(z_2)| &= \left|\int_I n(\xi)(\,\mathrm{d}F_\delta(\xi - z_1) - \,\mathrm{d}F_\delta(\xi - z_2)\right| \\ &= \left|n(b)V(b) - n(a)V(a) - \int_I V(\xi)\,\mathrm{d}n(\xi)\right| \\ &\le \sup_{\xi\in I} |V(\xi)| \int_I |\,\mathrm{d}n(\xi)| \le L_F\,|z_1 - z_2|\,TV(n),\end{aligned}$$

where the last inequalities follow because $V(a) = V(b) = 0$ and

$$|V(\xi)| = \left|\int_{z_2}^{z_1} f_\delta(\xi - \sigma)\,\mathrm{d}\sigma\right| \le L_F |z_1 - z_2|.$$

The boundedness follows since

$$\begin{aligned}|N(z)| &= \left|\int_{\mathbb{R}} n(z+\xi) f_\delta(\xi)\,\mathrm{d}\xi\right| \le \|n\|_\infty \int_{\mathbb{R}} f_\delta(\xi)\,\mathrm{d}\xi \\ &= \|n\|_\infty\end{aligned}$$

being f_δ nonnegative. □

It should be noticed that if n is Lipschitz then the corresponding averaged nonlinearity N will be Lipschitz independently on the characteristic of F_δ, see [23]. Lemma A.1 says that in order to have N Lipschitz even when n is not Lipschitz, we can impose the conditions of F_δ being absolute continuous with bounded derivative. The next lemma is the key to the proof of Theorem 3.1.

Lemma A.2 *Suppose the signal $y : [0, p] \to \mathbb{R}$ has Lipschitz constant L_y. Introduce a constant $\tilde{y}$ satisfying*

$$\min_{s\in[0,p]} y(s) \le \tilde{y} \le \max_{s\in[0,p]} y(s).$$

Suppose that F_δ is absolutely continuous with bounded derivative: $L_F = \sup_{\xi\in\mathbb{R}} |f_\delta(\xi)| < \infty$. Then

$$\left|\int_0^p n(-y(s) + \delta(s))\,\mathrm{d}s - \int_0^p n(-\tilde{y} + \delta(s))\,\mathrm{d}s\right| \le 2L_F L_y TV(n) p^2,$$

where $TV(n)$ is the total variation of n.

PROOF. From the definition of F_δ we have

$$F_{-y+\delta}(\xi) = \frac{1}{p}\mu\left(\{s \in [0, p) : \ -y(s) + \delta(s) \le \xi\}\right),$$

so that

$$\begin{aligned}E &\triangleq \left|\int_0^p n(-y(s) + \delta(s))\,\mathrm{d}s - \int_0^p n(-\tilde{y} + \delta(s))\,\mathrm{d}s\right| \\ &= p\left|\int_{\mathbb{R}} n(\xi)\,\mathrm{d}F_{-y+\delta}(\xi) - \int_{\mathbb{R}} n(\xi)\,\mathrm{d}F_{-\tilde{y}+\delta}(\xi)\right|.\end{aligned}$$

By hypothesis

$$\tilde{y} - L_y p \le y(s) \le \tilde{y} + L_y p, \quad \forall s \in [0, p],$$

and thus it follows that for any $\xi \in \mathbb{R}$,

$$F_{-\tilde{y}+\delta}(\xi - L_y p) \le F_{-y+\delta}(\xi) \le F_{-\tilde{y}+\delta}(\xi + L_y p). \tag{A.1}$$

On the other hand, since $F_{-\tilde{y}+\delta}$ is nondecreasing,

$$F_{-\tilde{y}+\delta}(\xi - L_y p) \le F_{-\tilde{y}+\delta}(\xi) \le F_{-\tilde{y}+\delta}(\xi + L_y p). \tag{A.2}$$

By combining (A.1) and (A.2) and using that $F_{-\tilde{y}+\delta}(\xi) = F_\delta(\xi + \tilde{y})$ is Lipschitz and non-negative, we get

$$\begin{aligned}F_{-y+\delta}(\xi) - F_{-\tilde{y}+\delta}(\xi) &\le \\ F_{-\tilde{y}+\delta}(\xi + L_y p) - F_{-\tilde{y}+\delta}(\xi - L_y p) &\le \\ 2L_F L_y p.\end{aligned}$$

In an analogous way,

$$-2L_F L_y p \leq F_{-y+\delta}(\xi) - F_{-\tilde{y}+\delta}(\xi).$$

So we can write

$$F_{-y+\delta}(\xi) = F_{-\tilde{y}+\delta}(\xi) + V(\xi)$$

with $|V(\xi)| \leq 2L_F L_y p$ and thus

$$E = p\left|\int_{\mathbb{R}} n(\xi)\,\mathrm{d}V(\xi)\right|.$$

Since for $s \in [0,p]$ we have $|y(s)-\tilde{y}| \leq L_y p$ and $|\delta(s)| \leq M_\delta$,

$$V(\xi) = 0, \quad \forall\, \xi \notin [-\tilde{y} - L_y p - M_\delta, -\tilde{y} + L_y p + M_\delta] \triangleq S.$$

The function $V(\xi)$ is of bounded variation and continuous from the right, since it is the difference of two functions that satisfy both these properties. By hypothesis n is of bounded variation with total variation $TV(n)$ so we can integrate by parts [14]:

$$\int_{[a,b]} n(\xi)\,\mathrm{d}V(\xi) = n(b)V(b) - n(a)V(a) - \int_{[a,b]} V(\xi)\,\mathrm{d}n(\xi),$$

where right and left limits of n and V are used in order to cope with discontinuities. If $[a,b] \supset S$ then $V(a) = V(b) = 0$, and thus in general

$$E \leq p\left|\int_S V(\xi)\,\mathrm{d}n(\xi)\right| \leq 2p^2 L_F L_y TV(n),$$

which proves the lemma. □

Lemmas A.1 and A.2 are used to prove the following result.

Lemma A.3 *If the assumptions of Theorem 3.1 hold, then there exist constants $\bar{K}, \tilde{K} > 0$ such that*

$$\left|\int_0^p f_i(x,s)n_i(g_i(x,s)+\delta_i)\,\mathrm{d}s - \int_0^p f_i(w,s)N_i(g_i(w,s))\,\mathrm{d}s\right| \leq \bar{K}\int_0^p |x(s)-w(s)|\,\mathrm{d}s + \tilde{K}p^2.$$

PROOF. For this proof we need a Lipschitz constant for $x(t)$ on $[0,T]$. By our assumptions we have

$$|x(t)| = |x_0 + \int_0^t \left(f_0(x(s),s) + \sum_{i=1}^m f_i(x(s),s)n_i(g_i(x(s),s)+\delta_i(s))\right)\mathrm{d}s| \leq |x_0| + (1+mM_n)L_f\int_0^t |x(s)|\,\mathrm{d}s + mM_nTM_I$$

where we used $|f_i(x,s) - f_i(0,s) + f_i(0,s)| \leq L_f|x(s)| + |f_i(0,s)|$ and introduced

$$M_I = \max_{i=1,\dots,m}\max_{t\in[0,T]} |f_i(0,t)|.$$

Grönvall-Bellman Lemma gives

$$|x(t)| \leq (mM_nM_IT + |x_0|)\,\mathrm{e}^{(1+mM_n)L_fT} =: M_x,\ t \in [0,T]. \tag{A.3}$$

This implies that $|\dot{x}(t)| \leq L_f M_x + mM_fM_n$ a.e., with

$$M_f = L_f(M_x + T) + \max_i |f_i(0,0)|. \tag{A.4}$$

This gives the Lipschitz bound $L_x = L_f M_x + mM_fM_n$. Hence, for any $\tilde{t} \in [0,p]$ and $\tilde{x} := x(\tilde{t})$, we have that $|x(s) - \tilde{x}| \leq L_x p$ for all $s \in [0,p]$.

Let us consider the following equality:

$$\begin{aligned}
&f_i(x,t)n_i(g_i(x,t)+\delta_i) - f_i(w,t)N_i(g_i(w,t)) \\
&= f_i(x,t)n_i(g_i(x,t)+\delta_i) - f_i(\tilde{x},\tilde{t})n_i(g_i(x,t)+\delta_i) && \text{(A.5a)}\\
&\quad + f_i(\tilde{x},\tilde{t})n_i(g_i(x,t)+\delta_i) - f_i(\tilde{x},\tilde{t})n_i(g_i(\tilde{x},\tilde{t})+\delta_i) && \text{(A.5b)}\\
&\quad + f_i(\tilde{x},\tilde{t})n_i(g_i(\tilde{x},\tilde{t})+\delta_i) - f_i(\tilde{x},\tilde{t})N_i(g_i(\tilde{x},\tilde{t})) && \text{(A.5c)}\\
&\quad + f_i(\tilde{x},\tilde{t})N_i(g_i(\tilde{x},\tilde{t})) - f_i(x,t)N_i(g_i(x,t)) && \text{(A.5d)}\\
&\quad + f_i(x,t)N_i(g_i(x,t)) - f_i(w,t)N_i(g_i(w,t)). && \text{(A.5e)}
\end{aligned}$$

Integrating (A.5) leads to the inequality

$$\begin{aligned}
&\left|\int_0^p [f_i(x,s)n_i(g_i(x,s)+\delta_i) - f_i(w,s)N_i(g_i(w,s))]\,\mathrm{d}s\right| \\
&\leq M_nL_f(L_x+1)p^2 && \text{(A.6a)}\\
&+|f_i(\tilde{x},\tilde{t})|\left|\int_0^p \left[n_i(g_i(x,t)+\delta_i) - n_i(g_i(\tilde{x},\tilde{t})+\delta_i)\right]\mathrm{d}s\right| && \text{(A.6b)}\\
&+M_nL_f(L_x+1)p^2 + M_fL_NL_g(L_x+1)p^2 && \text{(A.6c)}\\
&+(M_nL_f + M_fL_NL_g)\int_0^p |x-w|\,\mathrm{d}s, && \text{(A.6d)}
\end{aligned}$$

where we used that the integral of (A.5c) is zero by the definition of the averaged nonlinearity in (3). The other terms follows from the following arguments. First note that

$$|f_i(x,t) - f_i(\tilde{x},\tilde{t})| \leq L_f(L_x+1)p,$$

over the interval $[0,p]$. This gives (A.6a). Similarly,

$$|g_i(x,t) - g_i(\tilde{x},\tilde{t})| \leq L_g(L_x+1)p$$

over the interval $[0,p]$. Thus by applying Lemma A.2 with $-y(s) = g_i(x(s),s)$ and $-\tilde{y} = g_i(\tilde{x},\tilde{t})$, it follows that (A.6b) is bounded by

$$2p^2 M_f L_F L_g(L_x+1)TV(n) \qquad \text{(A.7)}$$

where we used M_f in (A.4). For the remaining terms we use that the Lipschitz constant of $f_i N_i$ is

$$L[f_i N_i] \leq L_f M_n + M_f L_N$$

We use this to show that (A.5d) is bounded by $M_n L_f(L_x+1)p + M_f L_N L_g(L_x+1)p$ and (A.6c) follows. In analogous way we can show that the upper bound of (A.5e) is $M_n L_f|x-w| + M_f L_N L_g|x-w|$.

For any $p>0$, we have shown that

$$\left|\int_0^p [f_i(x,s)\cdot n_i(g_i(x,s)+\delta_i) - f_i(w,s)\cdot N_i(g_i(w,s))]\ \mathrm{d}s\right| \leq \bar{K}\int_0^p |x-w|\,\mathrm{d}s + \tilde{K}p^2,$$

with

$$\bar{K} = M_n L_f + M_f L_N L_g$$
$$\tilde{K} = M_n L_f(L_x+1) + 2M_f L_F L_g(L_x+1)TV(n) + M_n L_f(L_x+1) + M_f L_N L_g(L_x+1)$$

□

Now we can proceed by showing that the approximation error between the dithered and the averaged system can be arbitrarily small by increasing the dither frequency, as stated in the theorem.

Proof of Theorem 3.1. Consider the dithered system (1) and the averaged system (2) on the time interval $[0,T]$ with $w_0 = x_0$. By integrating the right-hand sides of (1) and (2), we can write

$$|x(t)-w(t)| \leq \int_0^t |f_0(x,s) - f_0(w,s)|\,\mathrm{d}s + \sum_{i=1}^m \left|\int_0^t [f_i(x,s)n_i(g_i(x,s)+\delta_i) - f_i(w,s)N_i(g_i(w,s))]\ \mathrm{d}s\right|$$

for all $t \in [0,T]$.

If we introduce $\ell = \lfloor T/p \rfloor$, the largest integer such that $\ell p \leq T$, then by using the periodicity of δ_i,

$$|x(t)-w(t)| \leq \int_0^t |f_0(x(s),s) - f_0(w(s),s)|\,\mathrm{d}s + \sum_{k=0}^{\ell-1}\sum_{i=1}^m \left|\int_{kp}^{(k+1)p} f_i(x(s),s)n_i(g_i(x(s),s)+\delta_i(s))\,\mathrm{d}s - \int_{kp}^{(k+1)p} f_i(w(s),s)N_i(g_i(w(s),s))\,\mathrm{d}s\right| + V_1(p), \qquad \text{(A.8)}$$

where the last term is bounded as $|V_1(p)| \leq 2mM_f M_n p$.

The Lipschitz property of f_0 gives

$$\left|\int_0^t [f_0(x(s),s) - f_0(w(s),s)]\ \mathrm{d}s\right| \leq L_f \int_0^t |x(s)-w(s)|\ \mathrm{d}s.$$

Next we notice that each integral in the sum of (A.8) can be written as

$$\int_0^p f_i(x_k(s),s_k)n_i(g_i(x_k(s),s_k)+\delta_i(s))\,\mathrm{d}s - \int_0^p f_i(w_k(s),s_k)N_i(g_i(w_k(s),s_k))\,\mathrm{d}s$$

where the subscript k denotes a time translation: $s_k = s + kp$, $x_k(s) = x(s+kp)$, and similarly for w. Then applying Lemma A.3, each integral $\int_0^p (f_i(x_k,s_k)n_i(g_i(x_k,s_k)+\delta_i(s)))\,\mathrm{d}s$ can be approximated by $\int_0^p f_i(w_k,s_k)N_i(g_i(w_k,s_k))\,\mathrm{d}s$. Indeed, the Lipschitz assumptions on f_i and g_i are uniform in t so Lemma A.3 can be applied to all functions x_k. The approximation error has an upper bound $\bar{K}\int_0^p |x_k - w_k|\,\mathrm{d}s + \tilde{K}p^2$. By summing all the contributions given by the time intervals $[kp,(k+1)p] \subset [0,T]$, we get

$$|x(t)-w(t)| \leq K\int_0^t |x(s)-w(s)|\,\mathrm{d}s + m\tilde{K}pT + V_1(p) + V_2(p), \quad \forall t \in [0,T],$$

where $K = L_f + m\bar{K} = L_f + M_n L_f + M_f L_N L_g$ and $V_2(p)$ is bounded by

$$|V_2(p)| \leq m\bar{K}(M_x + M_w)p$$

where M_x was derived in (A.3) and M_w can in similar way be shown to be bounded by the same constant since $w(0) = x(0)$ and $\|N\|_\infty \leq \|n\|_\infty$, so that in fact $M_w \leq M_x$.

By applying Grönvall-Bellman Lemma [15], the theorem follows since

$$|x(t) - w(t)| \leq (m\tilde{K}Tp + V_1(p) + V_2(p)) \, \mathrm{e}^{KT} \quad \forall t \in [0, T]$$

where the right hand side is of order p. We have proven the bound in (5) with

$$c(\mathcal{K}, T) = m \sup_{x_0 \in \mathcal{K}} (\tilde{K}T + 2M_f M_n + 2\bar{K}M_x) \, \mathrm{e}^{KT}$$

where $\mathcal{K}$, $\bar{K}$, K, M_x and M_f all depend on x_0. □

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